SEMESTER – V, PAPER – VI-(A)

ELECTIVE–VI-(A); NUMERICAL METHODS

UNIT- I: **Finite Differences and Interpolation with Equal intervals**

1.Introduction, Finite Differences, Forward differences, Backward differences, Central Differences, Symbolic relations, n th Differences of some functions.

2.Advancing Differences formula, Differences of Factorial polynomials, Summation of series.

3.Newton's formulae for interpolation. Central Difference Interpolation Formulae.

UNIT – II: **Interpolation with Equal and Unequal intervals**

1.Gauss's central difference formulae, Sterling's central difference formula, Bessel's Formula.

2. Interpolation with unevenly spaced points, Divided differences and their properties, Newton's divided differences formula.

3.Lagranges's interpolation formula, Lagrange's Inverse interpolation formula.

UNIT – III: Numerical Differentiation

1. Derivatives using Newton's forward difference formula, Newton's backward difference formula.

2. Derivatives using central difference formula, sterling's interpolation formula.

3.Newton's divided difference formula, Maximum and minimum values of a tabulated function.

UNIT – IV: Numerical Integration

1.General quadrature formula on errors, Trapezoidal rule.

2.Simpson's 1/3 – rule, Simpson's 3/8 – rule, and Weddle's rules.

3.Euler – Maclaurin Formula of summation and quadrature, The Euler transformation.

UNIT – V: Numerical solution of ordinary differential equations

1.Introduction, Solution by Taylor's Series.

2.Picard's method of successive approximations.

3.Euler's method, Modified Euler's method, Runge – Kutta methods.

Reference Books :

1. Numerical Analysis by S.S.Sastry, published by Prentice Hall of India Pvt. Ltd., New Delhi. (Latest Edition)
2. Numerical Analysis by G. Sankar Rao published by New Age International Publishers, New – Hyderabad.
3. Finite Differences and Numerical Analysis by H.C Saxena published by S. Chand and Company, Pvt. Ltd., New Delhi.
4. Numerical methods for scientific and engineering computation by M.K.Jain, S.R.K.Iyengar, R.K. Jain.

Suggested Activities:

Seminar/ Quiz/ Assignments

1. Finite differences and Interpolation with Equal intervals

Let $y = f(x)$ be a function. Here x is called independent variable and y is dependent variable. The independent variable x is called as argument and y is called as entry. In $y = f(x)$ where $x \in [a\ b]$

Consider the arguments $x = x_0, x_1, x_2, \ldots x_n$ where $x_0 = a, x_1 = a + h, x_2 = a + 2h, \ldots..$

$x_n = a + nh$ then the corresponding values of y are $y_0, y_1, y_2, \ldots\ldots y_n$ here h is known as interval of differencing. Generally there are 3 types of differences:

i) Forward differences ii) Backward differences iii) Central differences.

Forward differences: - Let $y = f(x)$ be a function. If $x = x_0, x_1, x_2, \ldots x_n$ where $x_0 = a, x_1 = a + h, x_2 = a + 2h, \ldots\ldots, x_n = a + nh$ are the consecutive values of x then the corresponding values of y are $y_0, y_1, y_2, \ldots\ldots y_n$.

The differences $y_1 - y_0, y_2 - y_1, y_3 - y_2, \ldots$ are called the first order forward differences of the function $f(x)$

at the points $x = x_0, x_1, x_2, \ldots$ respectively and these are denoted by $\Delta y_0, \Delta y_1, \Delta y_2, \ldots$ respectively. The differences $\Delta y_1 - \Delta y_0, \Delta y_2 - \Delta y_1, \Delta y_3 - \Delta y_2, \ldots$ are called second order forward differences of the function $f(x)$ at the points $x = x_0, x_1, x_2, \ldots$ respectively and these are denoted by $\Delta^2 y_0, \Delta^2 y_1, \Delta^2 y_2, \ldots$ respectively.

Similarly we can define third order, fourth order… forward differences of $f(x)$ where Δ is the forward differences operator.

Forward difference operator: The operator Δ defined by $\Delta f(x) = f(x + h) - f(x)$ is called **forward difference operator (or) descending difference operator**.

Backward differences: - Let $y = f(x)$ be a function. If $x = x_0, x_1, x_2, \ldots x_n$ where $x_0 = a, x_1 = a + h, x_2 = a + 2h, \ldots\ldots, x_n = a + nh$ are the consecutive values of x then the corresponding values of y are $y_0, y_1, y_2, \ldots\ldots y_n$.

The differences $y_1 - y_0, y_2 - y_1, y_3 - y_2, \ldots$ are called the first order backward differences of the function $f(x)$

at the points $x = x_0, x_1, x_2, \ldots$ respectively and these are denoted by $\nabla y_0, \nabla y_1, \nabla y_2, \ldots$ respectively. The differences $\nabla y_1 - \nabla y_0, \nabla y_2 - \nabla y_1, \nabla y_3 - \nabla y_2, \ldots$ are called second order backward differences of the function $f(x)$ at the points $x = x_0, x_1, x_2, \ldots$ respectively and these are denoted by $\nabla^2 y_0, \nabla^2 y_1, \nabla^2 y_2, \ldots$ respectively.

Similarly we can define third order, fourth order… backward differences of $f(x)$ where ∇ is the backward differences operator.

Backward difference operator: The operator ∇ defined by $\nabla f(x) = f(x) - f(x - h)$ is called **backward difference operator (or) ascending difference operator**.

Central differences: - Let $y = f(x)$ be a function. If $x = x_0, x_1, x_2, \ldots x_n$ where $x_0 = a, x_1 = a + h, x_2 = a + 2h, \ldots\ldots, x_n = a + nh$ are the consecutive values of x then the corresponding values of y are $y_0, y_1, y_2, \ldots\ldots y_n$.

The differences $y_1 - y_0, y_2 - y_1, y_3 - y_2, \ldots$ are called the first order central differences of the function $f(x)$

at the points $x = x_0, x_1, x_2, \ldots$ respectively and these are denoted by $\delta y_{1/2}, \delta y_{3/2}, \delta y_{5/2}, \ldots$ respectively. The differences $\delta y_{3/2} - \delta y_{1/2}, \delta y_{5/2} - \delta y_{3/2}, \delta y_{7/2} - \delta y_{5/2}, \ldots$ are called second order central differences of the function $f(x)$ at the points $x = x_0, x_1, x_2, \ldots$ respectively and these are denoted by $\delta^2 y_1, \delta^2 y_2, \delta^2 y_3, \ldots$ respectively.

Similarly we can define third order, fourth order… Central differences of $f(x)$ where δ is the central differences operator.

Central difference operator: The operator δ defined by $\delta f(x) = f(x + h/2) - f(x - h/2)$ is called **central difference operator. i.e.** $\delta = E^{1/2} - E^{-1/2}$ $\quad (\because Ef(x) = f(x+h))$

Shift operator: -The Shift operator E defined by the rule $Ef(x) = f(x+h)$ where h is an increment

$$\therefore Ef(x) = f(x+h)$$

Average operator: -The Average operator μ is defined as $\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right]$

$$\mu f(x) = \frac{1}{2}\left[E^{\frac{1}{2}} f(x) + E^{-\frac{1}{2}} f(x)\right]$$

$$\mu = \frac{1}{2}\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right]$$

Average operator is also called the mean operator

Relationship between operators: -

1. Show that $\Delta = E - 1$ *or* $E = 1 + \Delta$

Sol) For any arbitrary function $f(x)$ we have show that $\Delta f(x) = (E-1)f(x)$

Now $\Delta f(x) = f(x+h) - f(x) = Ef(x) - f(x) = (E-1)f(x)$

$$\therefore \Delta = E - 1$$

2. Show that $\nabla = 1 - E^{-1}$

Sol) For any arbitrary function $f(x)$ we have show that $\nabla f(x) = (1 - E^{-1})f(x)$

$\nabla f(x) = f(x) - f(x-h) = f(x) - E^{-1}f(x) = (1 - E^{-1})f(x)$

$\therefore \nabla = 1 - E^{-1}$ or $E^{-1} = 1 - \nabla$

3. Show that $E = e^{hD} = 1 + \Delta$ where D is the differential operator.

Sol) For any arbitrary function $f(x)$ we have $Ef(x) = f(x+h)$ By Taylor's series

$$Ef(x) = f(x+h) = f(x) + hf^{1}(x) + \frac{h^2}{2!}f^{11}(x) + \frac{h^3}{3!}f^{111}(x) + \cdots$$

$$= 1.f(x) + hDf(x) + \frac{h^2}{2!}D^2f(x) + \frac{h^3}{3!}D^3f(x) + \cdots$$

$$= \left[1 + hD + \frac{h^2}{2!}D^2 + \frac{h^3}{3!}D^3 + \cdots\right] f(x)$$

$Ef(x) = e^{hD}f(x)$ $\quad \left(\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$

$\therefore E = e^{hD}$

Next $(1+\Delta)f(x) = f(x) + \Delta f(x) = f(x) + f(x+h) - f(x) = Ef(x)$

$\therefore 1 + \Delta = E$ Hence $E = e^{hD} = 1 + \Delta$

4. Show that $\Delta = 1 - e^{-hD}$ where D is the differential operator

Sol) we know that $E = e^{hD} \Rightarrow 1 + \Delta = e^{hD}$

$$\Rightarrow \frac{1}{1+\Delta} = \mathrm{e}^{-\mathrm{hD}} \Rightarrow (1+\Delta)^{-1} = \mathrm{e}^{-\mathrm{hD}} \Rightarrow 1 - \Delta = \mathrm{e}^{-\mathrm{hD}} \Rightarrow \Delta = 1 - \mathrm{e}^{-\mathrm{hD}}$$

5. Show that $1 + \delta^2\mu^2 = \left(1 + \frac{\delta^2}{2}\right)^2$ (OR) $1 + \frac{\delta^2}{2} = \sqrt{1 + \delta^2\mu^2}$

L.H.S = $1+\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^2 \left(\frac{E^{\frac{1}{2}}+E^{-\frac{1}{2}}}{2}\right)^2$

$$=1+\frac{1}{4}(E - E^{-1})^2 = 1 + \frac{1}{4}(E^2 - 2 + E^{-2}) = \frac{4+E^2-2+E^{-2}}{4} = \frac{1}{4}(E^2 + 2 + E^{-2}) = \frac{1}{4}(E + E^{-1})^2$$

$$= \left(\frac{E + E^{-1}}{2}\right)^2$$

R.H.S = $1 + \frac{\delta^2}{2} = 1 + \frac{1}{2}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^2 = 1 + \frac{1}{2}[E - 2 + E^{-1}] = \frac{2+E-2+E^{-1}}{2} = \frac{E+E^{-1}}{2}$

$$\left(1 + \frac{\delta^2}{2}\right)^2 = \left(\frac{E+E^{-1}}{2}\right)^2$$

$$\therefore 1 + \delta^2\mu^2 = \left(1 + \frac{\delta^2}{2}\right)^2 \ (OR)\ 1 + \frac{\delta^2}{2} = \sqrt{1 + \delta^2\mu^2}$$

6. $S.T$ $(i)\Delta = E\nabla$ $(ii)\nabla = E^{-1}\Delta$ $(iii)\Delta = \nabla(1-\nabla)^{-1}$ $(iv) 1 + \Delta = (E-1)\nabla^{-1}$ $(v)\Delta\nabla = \nabla\Delta = \delta^2$

$(vi)(1+\Delta)(1-\nabla) = 1$

$Sol(i)(E\nabla)f(\mathrm{x}) = \mathrm{E}(\nabla f(\mathrm{x})) = \mathrm{E}[f(\mathrm{x}) - f(\mathrm{x}-\mathrm{h})]$

$= \mathrm{E}f(\mathrm{x}) - \mathrm{E}f(x-h)$

$= f(x+h) - f(x-h+h) = f(x+h) - f(x) = \Delta f(\mathrm{x})$

$\therefore\ \Delta = E\nabla$

$(ii)\ (E^{-1}\Delta)f(x) = E^{-1}\big(\Delta f(x)\big) = E^{-1}[f(x+h) - f(x)] = E^{-1}f(x+h) - E^{-1}f(x)$

$= f(x+h-h) - f(x-h) = f(x) - f(x-h) = \nabla f(x)$

$\therefore\ \nabla = E^{-1}\Delta$

$(iii)\ \nabla(1-\nabla)^{-1} = \nabla(E^{-1})^{-1} \qquad since\ \nabla = 1 - E^{-1} \Longrightarrow E^{-1} = 1 - \nabla$

$= \nabla E = \Delta$

$(iv)(E-1)\nabla^{-1} = \Delta\nabla^{-1} \qquad since\ E - 1 = \Delta$

$= (E\nabla)\nabla^{-1} = E = 1 + \Delta$

$(v)\Delta\nabla = (\mathrm{E}-1)(1-\mathrm{E}^{-1}) = (\mathrm{E}-1-1+\mathrm{E}^{-1}) = (\mathrm{E}-2+\mathrm{E}^{-1}) = \left(\mathrm{E}^{\frac{1}{2}} - \mathrm{E}^{-\frac{1}{2}}\right)^2 = \delta^2$

Similarly we can prove $\nabla\Delta = \delta^2$

$(vi)(1+\Delta)(1-\nabla)f(x) = (1+\Delta)[f(x) - \nabla f(x)] = (1+\Delta)[f(x) - f(x) + f(x-h)]$

$= (1+\Delta)f(x-h) = f(x-h) + f(x) - f(x-h) = f(x)$

$\therefore\ (1+\Delta)(1-\nabla) = 1$

7. *Using the difference operator prove the following* $(i)\boldsymbol{\delta^2 E = \Delta^2}\ (ii)\boldsymbol{\mathrm{E}^{-\frac{1}{2}} = \mu - \frac{\delta}{2}}$

$iii)\ \mu^2 = 1 + \frac{1}{4}\delta^2$

Sol: (i)We know that $\delta = \mathrm{E}^{\frac{1}{2}} - \mathrm{E}^{-\frac{1}{2}} \Longrightarrow \delta^2 = \mathrm{E} - 2 + \mathrm{E}^{-1} = \frac{\mathrm{E}^2 - 2\mathrm{E} + 1}{\mathrm{E}}$

$\Longrightarrow \delta^2 E = (E-1)^2 = \Delta^2$

$(ii)\mu - \frac{\delta}{2} = \frac{\mathrm{E}^{\frac{1}{2}} + \mathrm{E}^{-\frac{1}{2}}}{2} - \frac{1}{2}\left[\mathrm{E}^{\frac{1}{2}} - \mathrm{E}^{-\frac{1}{2}}\right] = \frac{\mathrm{E}^{-\frac{1}{2}}}{2} + \frac{\mathrm{E}^{-\frac{1}{2}}}{2} = \mathrm{E}^{-\frac{1}{2}}$

$(iii)\ \mu^2 = \left[\frac{\mathrm{E}^{\frac{1}{2}} + \mathrm{E}^{-\frac{1}{2}}}{2}\right]^2 = \frac{1}{4}\left(\mathrm{E}^{\frac{1}{2}} + \mathrm{E}^{-\frac{1}{2}}\right)^2 = \frac{1}{4}\left[\left(\mathrm{E}^{\frac{1}{2}} - \mathrm{E}^{-\frac{1}{2}}\right)^2 + 4\right] = \frac{1}{4}(\delta^2 + 4) = 1 + \frac{1}{4}\delta^2$

8. P. T $\boldsymbol{f(a+nh) = f(a) + n_{c_1}\Delta f(a) + n_{c_2}\Delta^2 f(a) + \cdots + n_{c_n}\Delta^n f(a)}$

$Sol\colon f(a+nh) = E^n f(a) = (1+\Delta)^n f(a) \quad Since\ E = 1 + \Delta$

$= \left\{1 + n_{c_1}\Delta + n_{c_2}\Delta^2 + \cdots + n_{c_n}\Delta^n\right\}f(a)$

$$= f(a) + n_{c_1}\Delta f(a) + n_{c_2}\Delta^2 f(a) + \cdots + n_{c_n}\Delta^n f(a)$$

Fundamental theorem of differencial calculus: -The n^{th} order forward differences of an n^{th} degree polynomial $p_n(x)$ is a constant and $\Delta^{n+1}p_n(x) = 0$

Sol) Consider a polynomial $p_n(x)$ of n^{th} degree $p_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$

$$p_n(x+h) = a_0(x+h)^n + a_1(x+h)^{n-1} + a_2(x+h)^{n-2} + \cdots + a_{n-1(}x+h) + a_n$$

By def of Δ

$$\Delta p_n(x) = p_n(x+h) - p_n(x)$$

$$= a_0[(x+h)^n - x^n] + a_1[(x+h)^{n-1} - x^{n-1}] + \cdots + a_{n-1}[x+h-x] + (a_n - a_n)$$

$$= a_0\left[n_{c_1}x^{n-1}h + n_{c_2}x^{n-2}h^2 + \cdots + h^n\right] + a_1[(n-1)_{c_1}x^{n-2}h + (n-1)_{c_2}x^{n-3}h^2 + \cdots + h^{n-1}] + \cdots + a_{n-1}h$$

$\Delta p_n(x)$ is a polynomial of degree (n-1) with the coefficient of $x^{n-1}\ is\ a_0n_{c_1}h = a_0nh$

Similarly $\Delta^2 p_n(x)$ will be a polynomial of degree (n-2) with the coefficient of $x^{n-2}\ is\ = a_0n(n-1)h^2$

By repeating the process we can see that $\Delta^n p_n(x) = a_0n(n-1)(n-2)\ldots 3.2.1.h^n = a_0n!\,h^n$

Hence the n^{th} order forward differences of an n^{th} degree polynomial $p_n(x)$ is a constant

$$\therefore \Delta^n p_n(x) = constant$$

Hence $\quad \Delta^{n+1}p_n(x) = 0$

Note: $\Delta^k p_n(x) = \begin{cases} a_0n!\,h^n & if\ k = n \\ 0 & if\ k > n \end{cases}$ where $p_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$

Problems:

1. $\boldsymbol{If\ f(x) = k, k\ is\ constant\ then\ find\ (i)\Delta f(x)\ \ (ii)Ef(x)}$

$\boldsymbol{Sol}: f(x) = k \Longrightarrow f(x+h) = k$

$(i)\Delta f(x) = f(x+h) - f(x) = k - k = 0\ \ (ii)Ef(x) = f(x+h) = k$

2. $\boldsymbol{Evaluate\ (i)\ \Delta logx\ (ii)\Delta Tan^{-1}x}$

$Sol: (i)\Delta logx = log(x+h) - logx = log\left(\frac{x+h}{h}\right) = log\left(1 + \frac{x}{h}\right)$

$(ii)\Delta Tan^{-1}x = Tan^{-1}(x+h) - Tan^{-1}x\ = Tan^{-1}\left[\frac{x+h-x}{1+(x+h)x}\right] = Tan^{-1}\left[\frac{h}{1+(x+h)x}\right]$

3. $\boldsymbol{Evaluate\ \Delta^2(3e^x)}$

$Sol: \Delta^2(3e^x) = 3\Delta^2(e^x) = 3\Delta(\Delta e^x) = 3\Delta[e^{x+h} - e^x] = 3\Delta[e^x(e^h - 1)]$

$= 3(e^h - 1)\Delta e^x = 3(e^h - 1)[e^{x+h} - e^x] = 3e^x(e^h - 1)^2$

4. $\boldsymbol{Find\ the\ value\ E^2(x^2) when\ the\ value\ of\ x\ varing\ by\ a\ constant\ increment\ 2}$

$Sol: We\ have\ Ef(x) = f(x+h)\quad Given\ h = 2$

$E^2(x^2) = (x+2h)^2 = (x+4)^2 = x^2 + 8x + 16$

5. *Evaluate* $\boldsymbol{E^n(e^x)}$ ***when the interval differcing is h***

$Sol: We\ have\ Ef(x) = f(x+h)$

$E^n(e^x) = e^{x+nh}$

6. *Evaluate* $\boldsymbol{\Delta^3(1-x)(1-2x)(1-3x)}$***when the interval differencing being unity***

$Sol: We\ know\ that\ \Delta^k p_n(x) = \begin{cases} a_0 n!\, h^n & if\ k = n \\ 0 & if\ k > n \end{cases}$

$\Delta^3(1-x)(1-2x)(1-3x) = \Delta^3(1-6x^3) = (-6)(3!)h^3 = -36h^3 = -36$

7. *Evaluate* $\boldsymbol{\Delta^{10}(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)}$ ***when the interval differencing being unity***

$Sol: We\ know\ that\ \Delta^k p_n(x) = \begin{cases} a_0 n!\, h^n & if\ k = n \\ 0 & if\ k > n \end{cases}$

$\Delta^{10}(1-ax)(1-bx^2)(1-cx^3)(1-dx^4) = \Delta^{10}(abcdx^{10}) = (abcd)(10!)h^{10} = (abcd)(10!)$

8. *Show that* $\boldsymbol{e^x = \left(\frac{\Delta^2}{E}\right) e^x \cdot \frac{Ee^x}{\Delta^2 e^x}}$

$Sol: Let\ f(x) = e^x$

$We\ have\ \Delta f(x) = f(x+h) - f(x);\ \ Ef(x) = f(x+h) \Longrightarrow E(e^x) = e^{x+h}$

$\Delta^2(e^x) = \Delta(\Delta e^x) = \Delta[e^{x+h} - e^x] = \Delta[e^x(e^h - 1)]$

$= (e^h - 1)\Delta e^x = (e^h - 1)[e^{x+h} - e^x] = e^x(e^h - 1)^2$

$\left(\frac{\Delta^2}{E}\right) e^x = \Delta^2 E^{-1}(e^x) = \Delta^2(e^{x-h}) = e^{-h}\Delta^2(e^x) = e^{-h}(e^h - 1)^2 e^x$

$R.H.S = \left(\frac{\Delta^2}{E}\right) e^x \cdot \frac{Ee^x}{\Delta^2 e^x} = \frac{e^{-h}(e^h - 1)^2 e^x e^{x+h}}{e^x(e^h - 1)^2} = e^x = L.H.S$

9. *Find the values of* $\boldsymbol{(\Delta^2 E^{-1})U_x}$***and*** $\boldsymbol{(\Delta^2 U_x)(E^{-1}U_x)}$***are they equal?***

$Sol: (\Delta^2 E^{-1})U_x = \Delta^2 E^{-1}(U_x) = \Delta^2(U_{x-h}) = \Delta(\Delta(U_{x-h}) = \Delta[U_{x-h+h} - U_{x-h}]$

$= \Delta[U_x - U_{x-h}] = \Delta(U_x) - \Delta(U_{x-h}) = U_{x+h} - U_x - \{U_{x-h+h} - U_{x-h}\}$

$= U_{x+h} - U_x - \{U_x - U_{x-h}\} = U_{x+h} - 2U_x + U_{x-h}$

$\therefore (\Delta^2 E^{-1})U_x = U_{x+h} - 2U_x + U_{x-h}$

$\Delta^2 U_x = \Delta(\Delta U_x) = \Delta[U_{x+h} - U_x] = \Delta U_{x+h} - \Delta U_x = U_{x+h+h} - U_{x+h} - \{U_{x+h} - U_x\} = U_{x+2h} - 2U_{x+h} + U_x$

$\therefore \Delta^2 U_x = U_{x+2h} - 2U_{x+h} + U_x$

$E^{-1}(U_x) = U_{x-h}$

$(\Delta^2 U_x)(E^{-1}U_x) = [U_{x+2h} - 2U_{x+h} + U_x]U_{x-h}$

$\therefore (\Delta^2 E^{-1})U_x and\ (\Delta^2 U_x)(E^{-1}U_x)\ are\ not\ equal$

10. *Evaluate* $\boldsymbol{\frac{\Delta^2}{E}(x^3)}$***when*** $\boldsymbol{h = 1}$

$Sol: \frac{\Delta^2}{E}(x^3) = \frac{(E-1)^2}{E}(x^3) = \frac{(E^2 - 2E + 1)}{E}(x^3) = (E - 2 + E^{-1})(x^3)$

$= E(x^3) - 2(x^3) + E^{-1}(x^3) = (x+1)^3 - 2x^3 + (x-1)^3$

$= x^3 + 3x^2 + 3x + 1 - 2x^3 + x^3 - 3x^2 + 3x - 1$

$= 6x$

11. *Prove that* $\boldsymbol{u_x = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + \cdots + \Delta^{n-1} u_{x-n} + \Delta^n u_{x-n}}$

$Sol: Consider\ u_x - \Delta^n u_{x-n} = u_x - \Delta^n \cdot E^{-n} u_x = \left(1 - \frac{\Delta^n}{E^n}\right) u_x = \left(\frac{E^n - \Delta^n}{E^n}\right) u_x$

$= E^{-n}(E^n - \Delta^n) u_x = E^{-n}\left(\frac{E^n - \Delta^n}{1}\right) u_x \quad [\because \Delta = E - 1 \Longrightarrow 1 = E - \Delta\,]$

$= E^{-n}\left(\frac{E^n - \Delta^n}{E - \Delta}\right) u_x \quad since\ \left(\frac{x^n - y^n}{x - y}\right) = x^{n-1} + yx^{n-2} + y^2x^{n-3} + \cdots + y^{n-1}$

$= E^{-n}[E^{n-1} + \Delta E^{n-2} + \Delta^2 E^{n-3} + \cdots + \Delta^{n-1}] u_x$

$= [E^{-1} + \Delta E^{-2} + \Delta^2 E^{-3} + \cdots + \Delta^{n-1} E^{-n}] u_x$

$= E^{-1} u_x + \Delta E^{-2} u_x + \Delta^2 E^{-3} u_x + \cdots + \Delta^{n-1} E^{-n} u_x$

$\therefore u_x - \Delta^n u_{x-n} = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + \cdots + \Delta^{n-1} u_{x-n}$

$u_x = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + \cdots + \Delta^{n-1} u_{x-n} + \Delta^n u_{x-n}$

12. $P.T\ u_1 x + u_2 x^2 + u_3 x^3 + \cdots = \frac{x}{1-x} u_1 + \frac{x^2}{(1-x)^2} \Delta u_1 + \frac{x^3}{(1-x)^3} \Delta^2 u_1 + \cdots$

$Sol: u_1 x + u_2 x^2 + u_3 x^3 + \cdots = xEu_0 + x^2E^2u_0 + x^3E^3u_0 + \cdots = (xE + x^2E^2 + x^3E^3 + \cdots) u_0$

$= xE(1 + xE + x^2E^2 + x^3E^3 + \cdots) u_0 = xE(1 - xE)^{-1} u_0 = \frac{xE}{1 - xE} u_0 = \frac{xE}{1 - x(1+\Delta)} u_0$

$= \frac{xE}{1-x-x\Delta} u_0 = \frac{xE}{(1-x)\left[1 - \frac{x\Delta}{1-x}\right]} u_0 = \frac{x}{1-x}\left[1 - \frac{x\Delta}{1-x}\right]^{-1} Eu_0 \quad \text{since } (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

$= \frac{x}{1-x}\left[1 + \frac{x\Delta}{1-x} + \frac{x^2\Delta^2}{(1-x)^2} + \cdots\right] u_1 = \frac{x}{1-x} u_1 + \frac{x^2}{(1-x)^2} \Delta u_1 + \frac{x^3}{(1-x)^3} \Delta^2 u_1 + \cdots$

$\therefore u_1 x + u_2 x^2 + u_3 x^3 + \cdots = \frac{x}{1-x} u_1 + \frac{x^2}{(1-x)^2} \Delta u_1 + \frac{x^3}{(1-x)^3} \Delta^2 u_1 + \cdots$

13. $P.T\ u_{x+n} = u_n + x\Delta u_{n-1} + \frac{x(x+1)}{2!} \Delta^2 u_{n-2} + \frac{x(x+1)(x+2)}{3!} \Delta^3 u_{n-3} + \cdots$

$Sol: R.H.S = u_n + x\Delta u_{n-1} + \frac{x(x+1)}{2!} \Delta^2 u_{n-2} + \frac{x(x+1)(x+2)}{3!} \Delta^3 u_{n-3} + \cdots$

$= u_n + x\Delta E^{-1} u_n + \frac{x(x+1)}{2!} \Delta^2 E^{-2} u_n + \frac{x(x+1)(x+2)}{3!} \Delta^3 E^{-3} u_n + \cdots$

$= \left[1 + x\Delta E^{-1} + \frac{x(x+1)}{2!} \Delta^2 E^{-2} + \frac{x(x+1)(x+2)}{3!} \Delta^3 E^{-3} + \cdots\right] u_n$

$= [1 - \Delta E^{-1}]^{-x} u_n = [1 - (E-1)E^{-1}]^{-x} u_n = [1 - (1 - E^{-1})]^{-x} u_n = E^x u_n = u_{x+n}$

$\therefore u_{x+n} = u_n + x\Delta u_{n-1} + \frac{x(x+1)}{2!} \Delta^2 u_{n-2} + \frac{x(x+1)(x+2)}{3!} \Delta^3 u_{n-3} + \cdots$

14. $P.T\ y_{x+n} - n_{c_1} y_{x+n-1} + n_{c_2} y_{x+n-2} - \cdots + (-1)^n y_x = \Delta^n y_x$

$Sol: y_{x+n} - n_{c_1} y_{x+n-1} + n_{c_2} y_{x+n-2} - \cdots + (-1)^n y_x$

$= E^n y_x - n_{c_1} E^{n-1} y_x + n_{c_2} E^{n-2} y_x - \cdots + (-1)^n y_x$

$= [E^n - n_{c_1} E^{n-1} + n_{c_2} E^{n-2} - \cdots + (-1)^n] y_x = (E-1)^n y_x = \Delta^n y_x$

1. *Construct a forward difference table from the following data and evaluate* $\Delta^4 y_0$

x	0	1	2	3	4
y	1	1.5	2.2	3.1	4.6

Sol) Construct the difference table is given as follows

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		0.5			
1	1.5		0.2		
		0.7		0	
2	2.2		0.2		0.4
		0.9		0.4	
3	3.1		0.6		
		1.5			
4	4.6				

From the table $\Delta^4 y_0 = 0.4$

2. *Construct a backward difference table from the following data and evaluate* $\nabla^4 y_0$

x	10	20	30	40	50
y	1.0000	1.3010	1.4771	1.6021	1.6990

Sol) Construct the difference table is given as follows

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
10	1.0000				
		0.3010			
20	1.3010		-0.1249		
		0.1761		0.0738	
30	1.4771		-0.0511		-0.0508
		0.1250		0.0230	
40	1.6021		-0.0281		
		0.0969			
50	1.6990				

From the table $\nabla^4 y_0 = -0.0508$

$3. Find\ the\ missing\ term\ from\ the\ following\ data$

x	0	1	2	3	4
y	1	3	9	?	81

Sol) Construct the difference table is given as follows

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		2			
1	3		4		
		6		a-19	
2	9		a-15		124-4a
		a-9		105-3a	
3	a		90-2a		
		81-a			
4	81				

Here are given 4 values out of 5values so we have $\Delta^4 y_0 = 0 \Longrightarrow 124 - 4a = 0 \Longrightarrow a = \frac{124}{4} = 31$

2 nd Method:

Here are given 4 values out of 5values so we have $\Delta^4 y_0 = 0$

Since $\Delta^4 y_0 = 0 \Longrightarrow (E-1)^4 y_0 = 0 \Longrightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1)y_0 = 0$

$\Longrightarrow y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0 \Longrightarrow 81 - 4y_3 + 6(9) - 4(3) + 1 = 0 \Longrightarrow 124 - 4y_3 = 0 \Longrightarrow y_3 = 31$

4. *Find the missing term from the following data*

x	45	50	55	60
y	3.0	-	2.0	0.225

Sol) Construct the difference table is given as follows

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	3.0			
		a-3		
50	a		5-2a	
		2-a		3a-8.775
55	2.0		a-3.775	
		-1.775		
60	0.225			

Here are given 3 values out of 4values so we have $\Delta^3 y_0 = 0 \Longrightarrow 3a - 8.775 = 0 \Longrightarrow a = \frac{8.775}{3} = 2.925$

5. *Find the missing term from the following data*

x	0	5	10	15
y	7	11	-	18

Sol) Construct the difference table is given as follows

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	7			
		4		
5	11		a-15	
		a-11		44-3a
10	a		29-2a	
		18-a		
15	18			

Here are given 3 values out of 4values so we have $\Delta^3 y_0 = 0 \Longrightarrow 44 - 3a = 0 \Longrightarrow a = \frac{44}{3} = 14.7$

6. *Find the missing term from the following data*

x	0	5	10	15
y	6	10	-	17

Sol) Construct the difference table is given as follows

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	6			
		4		
5	10		a-14	
		a-10		41-3a
10	a		27-2a	
		17-a		
15	17			

Here are given 3 values out of 4values so we have $\Delta^3 y_0 = 0 \Longrightarrow 41 - 3a = 0 \Longrightarrow a = \frac{41}{3} = 13.7$

$7.\,Find\ the\ missing\ term\ from\ the\ following\ data$

x	1	2	3	4	5	6	7
y	2	4	8	?	32	64	128

Sol) Construct the difference table is given as follows

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1	2						
		2					
2	4		2				
		4		a-14			
3	8		a-12		66-4a		
		a-8		52-3a			
4	a		40-2a		6a-92	10a-158	
		32-a		3a-40			322-20a
5	32		a		72-4a	164-10a	
		32		32-a			
6	64		32				
		64					
7	128						

Here are given 6 values out of 7values so we have $\Delta^6 y_0 = 0 \Longrightarrow 322 - 20a = 0 \Longrightarrow a = \frac{322}{20} = 16.1$

$\boldsymbol{8.\ Find\ f(2)\ if\ f(-1) = 2; f(0) = 1; f(1) = 0; f(3) = -1}$

Sol) Construct the difference table is given as follows

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-1	2				
		-1			
0	1		0		
		-1		a+1	
1	0		a+1		-3-4a
		a		-2-3a	
2	a		-1-2a		
		-1-a			
3	-1				

Here are given 4 values out of 5values so we have $\Delta^4 y_0 = 0 \Longrightarrow -3 - 4a = 0 \Longrightarrow a = \frac{-3}{4}$

$f(2) = -\frac{3}{4}$

9. $If\ u_1 = 1;\ u_3 = 17;\ u_4 = 43; and\ u_5 = 89\ find\ the\ value\ of\ u_2$

Sol) Construct the difference table is given as follows

x	u	Δu	$\Delta^2 u$	$\Delta^3 u$	$\Delta^4 u$
1	1				
		a-1			
2	a		18-2a		
		17-a		3a-9	
3	17		a+9		20-4a
		26		11-a	
4	43		20		
		46			
5	89				

Here are given 4 values out of 5values so we have $\Delta^4 y_0 = 0 \Longrightarrow 20 - 4a = 0 \Longrightarrow a = 5$
$\boldsymbol{u_2} = 5$

10. *Find the missing term from the following data*

x	0	1	2	3	4	5
y	0	-	8	15	-	35

Sol) Construct the difference table is given as follows

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	0				
		a			
1	a		8-2a		
		8-a		3a-9	
2	8		a-1		-4a+b-12
		7		-a+b-21	
3	15		b-22		a-4b+93
		b-15		72-3b	
4	b		50-2b		
		35-b			
5	35				

Here are given 4 values out of 6values so we have $\Delta^4 y_0 = 0 \Longrightarrow -4a + b - 12 = 0 \; and \; a - 4b + 93 = 0$
$Solving \; above \; equations \; we \; get$
$\Longrightarrow -16a + 4b - 48 = 0$

$\Longrightarrow a - 4b + 93 = 0 \; adding \; we \; get - 15a + 45 = 0 \Longrightarrow a = 3$

$and \; from \; second \; equ \; 3 - 4b + 93 = 0 \Longrightarrow 4b = 96 \Longrightarrow b = 24$

$$\therefore \; y_1 = 3 \; and \; y_4 = 24$$

Factorial function: $The \; factorial \; function \; x^{(n)} \; is \; defined \; as \; x^{(n)} = x(x-h)(x-2h) \ldots$

$(x-(n-1)h)$ *for* $n = 1, 2, 3 \ldots$ *and* $x^{(0)}$ *is defined as* 1. *If the interval of differencing is* 1 *then*

$x^{(n)} = x(x-1)(x-2)\ldots\big(x-(n-1)\big)$ *In particular* $x^{(1)} = x$; $x^{(2)} = x(x-1)$; $x^{(3)} = x(x-1)(x-2)$

1. Expreess $f(x) = x^4 - 4x^3 + 7x^2 + 3x - 6$ in terms of the factorial notations

Sol: *Let* $f(x) = x^4 - 4x^3 + 7x^2 + 3x - 6 = Ax^{(4)} + Bx^{(3)} + Cx^{(2)} + Dx^{(1)} + E$

By Synthetic division

1	**1**	**-4**	**7**	**3**	**-6 = E**
	0	**1**	**-3**	**4**	
2	**1**	**-3**	**4**	**7 = D**	
	0	**2**	**-2**		
3	**1**	**-1**	**2 = C**		
	0	**3**			
4	**1**	**2 = B**			
	0				
	1 = A				

$\therefore f(x) = x^{(4)} + 2x^{(3)} + 2x^{(2)} + 7x^{(1)} - 6x^{(0)}$

2. Expreess $f(x) = x^4 - 5x^3 + 3x + 4$ in terms of the factorial notations

Sol: *Let* $f(x) = x^4 - 5x^3 + 3x + 4 = Ax^{(4)} + Bx^{(3)} + Cx^{(2)} + Dx^{(1)} + E$

By Synthetic division

1	**1**	**-5**	**0**	**3**	**4 = E**
	0	**1**	**-4**	**-4**	
2	**1**	**-4**	**-4**	**-1 = D**	
	0	**2**	**-4**		
3	**1**	**-2**	**-8 = C**		
	0	**3**			
4	**1 = A**	**1 = B**			

$\therefore f(x) = x^{(4)} + x^{(3)} - 8x^{(2)} - x^{(1)} + 4x^{(0)}$

3. *Find the function whose first differeces is* $9x^2 + 11x + 5$

***Sol*:** *Let* $f(x)$ *be the required function*

Given $\Delta f(x) = 9x^2 + 11x + 5 = Ax^{(2)} + Bx^{(1)} + Cx^{(0)}$

By Synthetic division

$$\begin{array}{c|lll} 1 & 9 & 11 & \underline{5} = C \\ & 0 & 9 & \\ \hline 2 & 9 & \underline{20} = B & \\ & 0 & & \\ \hline & 9 = A & & \end{array}$$

$\therefore \Delta f(x) = 9x^{(2)} + 20x^{(1)} + 5x^{(0)}$ *Intergration on both side*

$$f(x) = 9\frac{x^{(3)}}{3} + 20\frac{x^{(2)}}{2} + 5x^{(1)} + c = 3x^{(3)} + 10x^{(2)} + 5x^{(1)} + c$$

$$= 3x(x-1)(x-2) + 10x(x-1) + 5x + c = 3x^3 + x^2 + x + c$$

4. *Find the function whose first differeces is* $x^3 + 4x^2 + 9x + 12$

***Sol*:** *Let* $f(x)$ *be the required function*

Given $\Delta f(x) = x^3 + 4x^2 + 9x + 12 = Ax^{(3)} + Bx^{(2)} + Cx^{(1)} + D$

By Synthetic division

$$\begin{array}{c|llll} 1 & 1 & 4 & 9 & \underline{12} = D \\ & 0 & 1 & 5 & \\ \hline 2 & 1 & 5 & \underline{14} = C & \\ & 0 & 2 & & \\ \hline & 1 = A & 7 = B & & \end{array}$$

$\therefore \Delta f(x) = x^{(3)} + 7x^{(2)} + 14x^{(1)} + 12$ *Intergration on both side*

$$f(x) = \frac{x^{(4)}}{4} + 7\frac{x^{(3)}}{3} + 14\frac{x^{(2)}}{2} + 12x^{(1)} + c = \frac{x^{(4)}}{4} + 7\frac{x^{(3)}}{3} + 7x^{(2)} + 12x^{(1)} + c$$

$$= \frac{1}{4}x(x-1)(x-2)(x-3) + \frac{7}{3}x(x-1)(x-2) + 7x(x-1) + 12x + c$$

$$= \frac{1}{4}(x^4 - 6x^3 + 11x^2 - 6x) + \frac{7}{3}(x^3 - 3x^2 + 2x) + 7(x^2 - x) + 12x + c$$

$= \frac{x^4}{4} + \left(-\frac{6}{4} + \frac{7}{3}\right)x^3 + \left(\frac{11}{4} - 7 + 7\right)x^2 + \left(-\frac{6}{4} + \frac{14}{3} - 7 + 12\right)x + c = \frac{x^4}{4} + \frac{5x^3}{6} + \frac{11x^2}{4} + \frac{49x}{6} + c$

Interpolation with equal intervals

Suppose $y = f(x)$ be a function and x takes the values $a, a+h, a+2h, \dots, a+nh$ then y takes the values $f(a), f(a+h), f(a+2h), \dots, f(a+nh)$.

Interpolation: - The process of computing the values of the function inside range of the variables is called interpolation.

Extrapolation: - The process of computing the values of the function outside range of the variables is called extrapolation.

Newton's forward interpolation with equal intervals: -If $y = f(x)$ is a function and x takes the values $a, a+h, a+2h, \dots, a+nh$ then y takes the value $f(a), f(a+h), f(a+2h), \dots, f(a+nh)$ then

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \cdots + \frac{u(u-1)\dots(u-(n-1))}{n!}\Delta^n f(a) \quad where\ u = \frac{x-x_0}{h}$$

Proof) Define a polynomial $f(x) = A_0 + A_1(x-a) + A_2(x-a)(x-a-h) + A_3(x-a)(x-a-h)(x-a-2h) + \cdots + A_n(x-a)(x-a-h)\dots(x-a-(n-1)h) \to (1)$ $where\ A_0, A_1, \dots, A_n$ are constants

Put $x = a\ in\ (1)$ we get $f(a) = A_0$

Put $x = a+h\ \ in\ (1)$we get $f(a+h) = A_0 + A_1 h \Rightarrow f(a+h) = f(a) + A_1 h \Rightarrow A_1 = \frac{1}{h}[f(a+h) - f(a)]$

$$\Rightarrow A_1 = \frac{1}{h}\Delta f(a)$$

Put $x = a+2h\ \ in\ (1)$ we get $f(a+2h) = A_0 + A_1(2h) + A_2(2h)(h)$

$$\Rightarrow 2h^2 A_2 = f(a+2h) - f(a) - \frac{2h}{h}\Delta f(a)$$

$$= f(a+2h) - 2\Delta f(a) - f(a)$$

$$= f(a+2h) - 2[f(a+h) - f(a)] - f(a)$$

$$= f(a+2h) - 2f(a+h) + f(a) = \Delta^2 f(a)$$

$$\therefore A_2 = \frac{1}{2!h^2}\Delta^2 f(a)$$

Proceeding in this way by putting $x = a+3h, \dots, a+nh$ we get $A_3 = \frac{1}{3!h^3}\Delta^3 f(a) \dots \dots A_n = \frac{1}{n!h^n}\Delta^n f(a)$

Substituting the values of $A_0, A_1, A_2, \dots, A_n\ in\ (1)$ we get

$f(x) = f(a) + \frac{\Delta f(a)}{h}(x-a) + \frac{\Delta^2 f(a)}{2!h^2}(x-a)(x-a-h) + \cdots + \frac{\Delta^n f(a)}{n!h^n}(x-a)\dots(x-a-(n-1)h) \to (2)$

Put $u = \frac{x-x_0}{h}$ $so\ that\ x = a + uh$ we get

$$f(a+uh) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \cdots + \frac{u(u-1)\dots(u-(n-1))}{n!}\Delta^n f(a)$$

$f(a+uh) = f(a) + u_{c_1}\Delta f(a) + u_{c_2}\Delta^2 f(a) + u_{c_3}\Delta^3 f(a) + \cdots + u_{c_n}\Delta^n f(a)$

(or)

$$y_x = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots + \frac{u(u-1)\dots(u-(n-1))}{n!}\Delta^n y_0$$

(or)

$f(a+nh) = f(a) + n_{c_1}\Delta f(a) + n_{c_2}\Delta^2 f(a) + \cdots + n_{c_n}\Delta^n f(a)$

Sol) $f(a+nh) = E^n f(a) = (1+\Delta)^n f(a)$ $(\because E = 1 + \Delta)$

$= [1 + n_{c_1}\Delta + n_{c_2}\Delta^2 + \cdots + n_{c_n}\Delta^n] f(a) = f(a) + n_{c_1}\Delta f(a) + n_{c_2}\Delta^2 f(a) + \cdots + n_{c_n}\Delta^n f(a)$

Newton's backward interpolation with equal intervals: - If $y = f(x)$ is a function and x takes the values $a, a+h, a+2h, \dots, a+nh$ then y takes the value $f(a), f(a+h), f(a+2h), \dots, f(a+nh)$ then

$$f(x) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a+nh) + \cdots + \frac{u(u+1)\dots(u+(n-1))}{n!}\nabla^n f(a+nh) \quad where\ \ x = a + nh + uh$$

Proof) Define a polynomial $f(x) = A_0 + A_1(x-a-nh) + A_2(x-a-nh)(x-a-(n-1)h) + \cdots + A_n(x-a-nh)(x-a-(n-1)h)\dots(x-a-h) \to (1)$ $where\ \ A_0, A_1, \dots, A_n$ are constants

Put $x = a + nh\ in\ (1)$ we get $f(a+nh) = A_0$

Put $x = a + (n-1)h\ \ in\ (1)$ we get $f(a+(n-1)h) = A_0 + A_1[a + (n-1)h - a - nh] = A_0 - A_1 h$

$\Rightarrow A_1 h = A_0 - f(a+(n-1)h) = f(a+nh) - f(a+(n-1)h) = \nabla f(a+nh)$

$\Rightarrow A_1 = \frac{1}{h}\nabla f(a+nh)$

Put $x = a + (n-2)h\ \ in\ (1)$ we get $f(a+(n-2)h) = A_0 + A_1[a+(n-2)h-a-nh] + A_2[a+(n-2)h-a-nh][a+(n-2)h-a-(n-1)h] = A_0 + A_1(-2h) + A_2(-2h)(-h)$

$\Rightarrow 2h^2 A_2 = f(a+(n-2)h) - f(a+nh) + 2\nabla f(a+nh) = \nabla^2 f(a+nh)$

$\therefore A_2 = \frac{1}{2!h^2}\nabla^2 f(a+nh)$ Proceeding in this way we get

$$A_3 = \frac{1}{3!h^3}\nabla^3 f(a+nh)$$

$$\vdots$$

$$A_n = \frac{1}{n!h^n}\nabla^n f(a+nh)$$

Substitute these values in (1) we get

$$f(x) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a+nh) + \cdots + \frac{u(u+1)\ldots(u+(n-1))}{n!}\nabla^n f(a+nh) \quad where\ \ x = a+nh+uh$$

Note: - This formula is particularly useful for interpolating the values of $f(x)$ near to the end of the set of given values.

1. *Compute $f(1.1)$ from the following data*

x	1	2	3	4	5
y	7	12	29	64	123

Sol) Construct the difference table is given as follows

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	7				
		5			
2	12		12		
		17		6	
3	29		18		0
		35		6	
4	64		24		
		59			
5	123				

$$Here\ a = 1\ h = 1, x = 1.1\ then\ \ u = \frac{x-a}{h} = \frac{1.1-1}{1} = 0.1$$

$By\ Newton's\ forward\ interpolation\ formula\ is$

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \cdots$$

$$\Longrightarrow f(1.1) = f(1) + 0.1\Delta f(1) + \frac{(0.1)(0.1-1)}{2!}\Delta^2 f(1) + \frac{0.1(0.1-1)(0.1-2)}{3!}\Delta^3 f(1) + 0$$

$$= 7 + 0.1(5) + \frac{(0.1)(0.1-1)}{2}(12) + \frac{0.1(0.1-1)(0.1-2)}{6}(6) + 0 = 7 + 0.5 - 0.54 + 0.171 = 7.131$$

$\boldsymbol{2. Compute\ y(1.6) from\ the\ following\ data}$

x	1	1.4	1.8	2.2
y	3.49	4.82	5.96	6.5

Sol) Construct the difference table is given as follows

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	3.49			
		1.33		
1.4	4.82		-0.19	
		1.14		-0.41
1.8	5.96		-0.6	
		0.54		
2.2	6.5			

$$Here\ a = 1\ h = 0.4, x = 1.6\ then\ \ u = \frac{x-a}{h} = \frac{1.6-1}{0.4} = 1.5$$

$By\ Newton's\ forward\ interpolation\ formula\ is$

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \cdots$$

$$\Longrightarrow f(1.6) = f(1) + 1.5\Delta f(1) + \frac{(1.5)(1.5-1)}{2!}\Delta^2 f(1) + \frac{1.5(1.5-1)(1.5-2)}{3!}\Delta^3 f(1)$$

$$= 3.49 + 1.5(1.33) + \frac{(1.5)(1.5-1)}{2}(-0.19) + \frac{1.5(1.5-1)(1.5-2)}{6}(-0.41)$$

$$= 3.49 + 1.995 - 0.07125 + 0.025625 = 5.439375 \cong 5.4394$$

$\boldsymbol{3. Find\ tan(0.17)\ from\ the\ following\ data}$

x	0.10	0.15	0.20	0.25	0.30	Ans:
$y = tanx$	0.1003	0.1511	0.2027	0.2553	0.3093	

4. $The\ population\ of\ a\ country\ in\ the\ decimal\ census\ were\ as\ under.\ Estimate\ the\ population\ for\ the\ year\ 1925\ from\ the\ following\ data$

$Year(x)$	1891	1901	1911	1921	1931
$Pop(y\ in\ thousand)$	46	66	81	93	101

Sol) Construct the difference table is given as follows

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

$Here\ a = 1931\ \ h = 10, x = 1925\ then\ \ u = \dfrac{x-a}{h} = \dfrac{1925-1931}{10} = -0.6$

$By\ Newton's\ Backward\ interpolation\ formula\ is$

$$f(x) = f(a) + u\nabla f(a) + \frac{u(u+1)}{2!}\nabla^2 f(a) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a) + \cdots$$

$$\Longrightarrow f(1925) = f(1931) - 0.6\nabla f(1931) + \frac{(-0.6)(-0.6+1)}{2!}\nabla^2 f(1931)$$

$$+ \frac{-0.6(-0.6+1)(-0.6+2)}{3!}\nabla^3 f(1931) + \frac{-0.6(-0.6+1)(-0.6+2)(-0.6+3)}{4!}\nabla^4 f(1931)$$

$$= 101 - 0.6(8) + \frac{(-0.6)(-0.6+1)}{2}(-4)$$

$$+\frac{-0.6(-0.6+1)(-0.6+2)}{6}(-1)+\frac{-0.6(-0.6+1)(-0.6+2)(-0.6+3)}{24}(-3)$$

$$= 101 - 4.8 + 0.48 + 0.1008 = 96.8368$$

$Hence, the\ population\ for\ 1925\ is\ 96.84\ thousand$

5. *Using Newton's backward formula find $f(0.7)$ from the following data.*

x	0.1	0.2	0.3	0.4	0.5	0.6
y	2.68	3.04	3.38	3.68	3.96	4.21

Sol) Construct the difference table is given as follows

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
0.1	2.68					
		0.36				
0.2	3.04		-0.02			
		0.34		-0.02		
0.3	3.38		-0.04		0.04	
		0.30		0.02		-0.07
0.4	3.68		-0.02		-0.03	
		0.28		-0.01		
0.5	3.96		-0.03			
		0.25				
0.6	4.21					

$$Here\ a = 0.6\ \ h = 0.1, x = 0.7\ then\ \ u = \frac{x-a}{h} = \frac{0.7-0.6}{0.1} = 1$$

$By\ Newton's\ Backward\ interpolation\ formula\ is$

$$f(x) = f(a) + u\nabla f(a) + \frac{u(u+1)}{2!}\nabla^2 f(a) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a) + \cdots$$

$$\Rightarrow f(0.7) = f(0.6) + 1\nabla f(0.6) + \frac{(1)(1+1)}{2!}\nabla^2 f(0.6) + \frac{1(1+1)(1+2)}{3!}\nabla^3 f(0.6)$$

$$+\frac{1(1+1)(1+2)(1+3)}{4!}\nabla^4 f(0.6)+\frac{1(1+1)(1+2)(1+3)(1+4)}{5!}\nabla^5 f(0.6)$$

$$=4.21+1(0.25)+\frac{(2)}{2}(-0.03)+\frac{6}{6}(-0.01)+\frac{24}{24}(-0.03)+\frac{120}{120}(-0.07)$$

$$=4.21+0.25-0.03-0.01-0.03-0.07=4.32$$

$Hence, f(0.7)=4.32$

6. *From the following table, find the number of students who obtain less than 45 marks*

$Marks(x)$	30-40	40-50	50-60	60-70	70-80
$No. of\ students(y)$	31	42	51	35	31

Sol) Construct the difference table is given as follows

$Marks\ less\ than\ x$	$No. of\ students\ y$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

$Here\ a=40\ \ h=10, x=45\ then\ \ u=\frac{x-a}{h}=\frac{45-40}{10}=\frac{5}{10}=0.5$

$By\ Newton's\ Forward\ interpolation\ formula\ is$

$$f(x)=f(a)+u\Delta f(a)+\frac{u(u-1)}{2!}\Delta^2 f(a)+\frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)+\cdots$$

$$\Rightarrow f(45)=f(40)+0.5\Delta f(40)+\frac{(0.5)(0.5-1)}{2!}\Delta^2 f(40)$$

$$+\frac{0.5(0.5-1)(0.5-2)}{3!}\Delta^3 f(40)+\frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!}\Delta^4 f(40)$$

$$=31+0.5(42)+\frac{(0.5)(0.5-1)}{2}(9)+\frac{0.5(0.5-1)(0.5-2)}{6}(-25)+\frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24}(37)$$

$$=31+21-1.125-1.563-1.445=47.867=48$$

Hence, the number of the students who obtain less than 45 *marks is* 48

7. *Find the cubic polynomial from the following data*

x	0	1	2	3	$Ans: f(x)=x^3-2x^2+1$
$y=f(x)$	1	0	1	10	

8. *Find the cubic polynomial from the following data*

x	0	1	2	3
$y=f(x)$	1	2	1	10

Sol) Construct the difference table is given as follows

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

$$Here\ a=0\ h=1, and\ \ u=\frac{x-a}{h}=\frac{x-0}{1}=x$$

By Newton's forward interpolation formula is

$$f(x)=f(a)+u\Delta f(a)+\frac{u(u-1)}{2!}\Delta^2 f(a)+\frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)+\cdots$$

$$\Longrightarrow f(x)=f(0)+x\Delta f(0)+\frac{x(x-1)}{2!}\Delta^2 f(0)+\frac{x(x-1)(x-2)}{3!}\Delta^3 f(0)$$

$$= 1 + x.1 + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12)$$

$$= 1 + x - x(x-1) + 2x(x-1)(x-2) = 1 + x - x^2 + x + 2x^3 - 2x^2 - 4x^2 + 4x$$

$$= 2x^3 - 7x^2 + 6x + 1 \ \textit{which is the required polynomial}$$

9. *Find the cubic polynomial passes through* $(0,1), (1,3), (2,7)$ *and* $(3,13)$

x	0	1	2	3
$y = f(x)$	1	3	7	13

Sol) Construct the difference table is given as follows

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		2		
1	3		2	
		4		0
2	7		2	
		6		
3	13			

Here $a = 0$ $h = 1,$ *and* $u = \frac{x-a}{h} = \frac{x-0}{1} = x$

By Newton's forward interpolation formula is

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \cdots$$

$$\Rightarrow f(x) = f(0) + x\Delta f(0) + \frac{x(x-1)}{2!}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{3!}\Delta^3 f(0)$$

$$= 1 + x.2 + \frac{x(x-1)}{2}(2) + \frac{x(x-1)(x-2)}{6}(0)$$

$$= 1 + 2x + x(x-1) = 1 + 2x + x^2 - x$$

$$= x^2 + x + 1 \ \textit{which is the required polynomial}$$

10. *Compute* $f(1.02)$ *from the following data*

x	1	1.1	1.2	1.3	1.4
y	0.841	0.891	0.932	0.964	0.985

Sol) Construct the difference table is given as follows

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	0.841				
		0.05			
1.1	0.891		-0.009		
		0.041		0	
1.2	0.932		-0.009		-0.002
		0.032		-0.002	
1.3	0.964		-0.011		
		0.021			
1.4	0.985				

$$Here\ a = 1\ h = 0.1, x = 1.02\ then\ \ u = \frac{x-a}{h} = \frac{1.02-1}{0.1} = 0.2$$

$$By\ Newton's\ forward\ interpolation\ formula\ is$$

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \cdots$$

$$\Rightarrow f(1.02) = f(1) + 0.2\Delta f(1) + \frac{(0.2)(0.2-1)}{2!}\Delta^2 f(1) + \frac{0.2(0.2-1)(0.2-2)}{3!}\Delta^3 f(1) + \frac{0.2(0.2-1)(0.2-2)(0.2-3)}{4!}\Delta^4 f(1)$$

$$= 0.841 + 0.2(0.05) + \frac{(0.2)(0.2-1)}{2}(-0.009) + \frac{0.2(0.2-1)(0.2-2)}{6}(0) + \frac{0.2(0.2-1)(0.2-2)(0.2-3)}{24}(-0.002)$$

$$= 0.841 + 0.01 + 0.00072 + 0.0000672 = 0.8517 \approx 0.852$$

2. Interpolation with Equal and Unequal intervals

Gauss's forward interpolation formula with equal intervals: - If $y = f(x)$ is a function, which takes the values … $y_{-2}, y_{-1}, y_0, y_1, y_2$ ... corresponding to the values of $x = \cdots x_0 - 2h, x_0 - h, x_0,\ x_0 + h, x_0 + 2h, \ldots$ then

$$y_x = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u+1)(u-2)}{4!}\Delta^4 y_{-2} + \cdots$$

Where $u = \frac{x-x_0}{h}$

Proof) The Newton's forward interpolation formula is given by

$$y_x = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots \quad \to (1)$$

We know $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1} \Rightarrow \Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$

Similarly, $\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1},\ \Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1}, \ldots etc.$

Also $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2},\ \Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}, \ldots etc.$

Substituting the values of $\Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0, \ldots .\ in\ (1)$ we get,

$$y_x = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{u(u-1)(u-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{u(u-1)(u-2)(u-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots$$

$$= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u-2)(u+1)}{4!}\Delta^4 y_{-1} + \cdots$$

$$= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u-2)(u+1)}{4!}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \cdots$$

$$y_x = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u+1)(u-2)}{4!}\Delta^4 y_{-2} + \cdots$$

Gauss's backward interpolation formula for equal intervals: - If $y = f(x)$ is a function, which takes the values … $y_{-2}, y_{-1}, y_0, y_1, y_2$ ... corresponding to the values of $x = \cdots x_0 - 2h, x_0 - h, x_0,\ x_0 + h, x_0 + 2h, \ldots$ then

$$y_x = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-2} + \frac{u(u+1)(u-1)(u+2)}{4!}\Delta^4 y_{-2} + \cdots.$$

Where $u = \frac{x-x_0}{h}$

Proof) The Newton's forward interpolation formula is given by

$$y_x = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots \quad \to (1)$$

We have $\Delta y_0 - \Delta y_{-1} = \Delta^2 y_{-1} \Longrightarrow \Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1}$, $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$, $\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}, \Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} \ldots . etc$

Also $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$ $\quad \Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} \ldots .. etc$

Substituting all these values in (1) we get

$$y_x = y_0 + u(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{u(u-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{u(u-1)(u-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{u(u-1)(u-2)(u-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots$$

$$y_x = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-1} + \cdots$$

$$y_x = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}(\Delta^3 y_{-2} + \Delta^4 y_{-2}) + \frac{(u+1)u(u-1)(u-2)}{4!}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \cdots$$

$$y_x = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-2} + \frac{u(u+1)(u-1)(u+2)}{4!}\Delta^4 y_{-2} + \cdots$$

Sterling's formula: - If $y = f(x)$ is a function, which takes the values $\ldots y_{-2}, y_{-1}, y_0, y_1, y_2 \ldots$ corresponding to the values of $x = \cdots x_0 - 2h, x_0 - h, x_0, \; x_0 + h, x_0 + 2h, \ldots$ then $y_x = y_0 + \frac{u}{2}(\Delta y_0 + \Delta y_{-1}) + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \cdots$

Proof) Taking the average of Gauss's forward and backward formula is called as sterling's formula.

We have Gauss's forward formula is

$$y_x = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u+1)(u-2)}{4!}\Delta^4 y_{-2} + \cdots \to (1)$$

Also, Gauss's backward formula is

$$y_x = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-2} + \frac{u(u+1)(u-1)(u+2)}{4!}\Delta^4 y_{-2} + \cdots \to (2)$$

Taking the average of (1) and (2), we get

$$y_x = \frac{(y_0 + y_0)}{2} + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{\Delta^2 y_{-1}}{2!}\left[\frac{u(u-1) + u(u+1)}{2}\right] + \left[\frac{u(u-1)(u+1)}{3!}\right]\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) +$$

$$y_x = y_0 + \frac{u}{2}(\Delta y_0 + \Delta y_{-1}) + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

Bessel's formula: - If $y = f(x)$ is a function, which takes the values $\ldots y_{-2}, y_{-1}, y_0, y_1, y_2 \ldots$ corresponding to the values of $x = \cdots x_0 - 2h, x_0 - h, x_0, \; x_0 + h, x_0 + 2h, \ldots$ then $y_x = \frac{y_0+y_1}{2} + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{u(u-1)\left(u-\frac{1}{2}\right)}{3!}\Delta^3 y_{-1} + \frac{u(u+1)(u-1)(u-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \cdots$

Proof) Gauss's forward formula is given by

$$y_x = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u+1)(u-2)}{4!}\Delta^4 y_{-2} + \cdots \to (1)$$

Also, Gauss's backward formula is

$$y_x = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{3!}\Delta^3 y_{-2} + \frac{u(u+1)(u-1)(u+2)}{4!}\Delta^4 y_{-2} + \cdots \to (2)$$

Starts with y_1 and replace u by $u-1$ and runs parallel to the Gauss's backward formula, we get

$$y_x = y_1 + (u-1)\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u-2)(u+1)}{4!}\Delta^4 y_{-1} + \cdots (3)$$

Taking the average of (1) and (3) we get

$$y_x = \frac{y_0+y_1}{2} + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{u(u-1)\left(u-\frac{1}{2}\right)}{3!}\Delta^3 y_{-1} + \frac{u(u+1)(u-1)(u-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \cdots$$

1. *Using Gauss forward formula find $f(3.75)$ from the following data.*

x	2.5	3.0	3.5	4.0	4.5	5.0
y	24.145	22.043	20.225	18.644	17.262	16.047

Sol: *Taking the origin at* $x = 3.5$ *i.e* $x_0 = 3.5$; $\; x = 3.75$; $\; h = 0.5$

$$u = \frac{x - x_0}{h} = \frac{3.75 - 3.5}{0.5} = \frac{0.25}{0.5} = 0.5$$

Construct the difference table is given as follows

x	$u=\frac{x-x_0}{h}$	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$	$\Delta^5 y_u$
2.5	-2	24.145					
			-2.102				
3.0	-1	22.043		0.284			
			-1.818		-0.047		
3.5	0	$\boxed{20.225}$		0.237		0.009	
			-1.581		-0.038		-0.003
4.0	1			0.199		0.006	
		18.644					
			-1.382		-0.032		
4.5	2			0.167			
		17.262					
			-1.215				
5.0	3						
		16.047					

F

The Gauss forward formula is $y_x = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1}$

$$+\frac{u(u-1)(u+1)(u-2)}{4!}\Delta^4 y_{-2} + \frac{u(u-1)(u+1)(u-2)(u+2)}{5!}\Delta^5 y_{-2} + \cdots$$

$$= 20.225 + 0.5(-1.581) + \frac{0.5(-0.5)}{2}(0.237) + \frac{0.5(-0.5)(1.5)}{6}(-0.038)$$

$$+\frac{0.5(-0.5)(1.5)(-1.5)}{24}(0.009) + \frac{0.5(-0.5)(1.5)(-1.5)(2.5)}{120}(-0.003)$$

$$= 20.225 - 0.7905 - 0.029625 + 0.00238 + 0.002109375 + 0.0002106 = 19.40 \text{ approximately}$$

2. Use Gauss's forward formula to find y_{30} given that $y_{21} = 18.4708$; $y_{25} = 17.8144$;

$y_{29} = 17.1070$; $y_{33} = 16.3432$; $y_{37} = 15.5154$

Sol: *Taking the origin at* $x = 29$ *i.e* $x_0 = 29$; $x = 30$; $h = 4$

$$u = \frac{x-x_0}{h} = \frac{30-29}{4} = \frac{1}{4} = 0.25$$

Construct the difference table is given as follows

x	$u=\dfrac{x-x_0}{h}$	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$
21	-2	18.4708				
			-0.6554			
25	-1	17.8144		-0.0510		
			-0.7074		-0.0054	
29	0	17.1070		-0.0564		-0.0022
			-0.7638		-0.0076	
33	1			-0.0640		
		16.3432	-0.8278			
37	2					
		15.5154				

$$\textit{The Gauss forward formula is}\ \ y_x = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1}$$

$$+\frac{u(u-1)(u+1)(u-2)}{4!}\Delta^4 y_{-2} + \cdots$$

$$y_{30} = 17.1070 + 0.25(-0.7638) + \frac{0.25(0.25-1)}{2}(-0.0564) + \frac{0.25(0.25-1)(0.25+1)}{6}(-0.0076)$$

$$+\frac{0.25(0.25-1)(0.25+1)(0.25-2)}{24}(-0.0022)$$

$$= 17.1070 - 0.19095 + 0.0052875 + 0.0002968 - 0.0000375 = 16.9216\ \ \textit{approximately}$$

$3.$ *Use Gauss' s forward formula to find* y_{32} *given that* $y_{25} = 0.2707;\ y_{30} = 0.3027;$

$y_{35} = 0.3386;\ y_{40} = 0.3794$

Sol: *Taking the origin at* $x = 30$ *i.e* $x_0 = 30;\ \ x = 32;\ \ h = 5$

$$u = \frac{x-x_0}{h} = \frac{32-30}{5} = \frac{2}{5} = 0.4$$

Construct the difference table is given as follows

x	$u = \dfrac{x - x_0}{h}$	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$
25	-1	0.2707			
			0.032		
30	0	0.3027		0.0039	
			0.0359		0.001
35	1	0.3386		0.0049	
			0.0408		
40	2	0.3794			

The Gauss forward formula is $y_x = y_0 + u\Delta y_0 + \dfrac{u(u-1)}{2!}\Delta^2 y_{-1} + \dfrac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1}$

$+ \dfrac{u(u-1)(u+1)(u-2)}{4!}\Delta^4 y_{-2} + \cdots$

$y_{32} = 0.3027 + 0.4(0.0359) + \dfrac{0.4(0.4-1)}{2}(0.0039) + \dfrac{0.4(0.4-1)(0.4+1)}{6}(0.001)$

$= 0.3027 + 0.01436 - 0.000468 - 0.000056 = 0.3167$ *approximately*

4. *Using Gauss forward formula find* $e^{1.17}$ ***from the following data.***

x	1.00	1.05	1.10	1.15	1.20	1.25
$y = e^x$	2.7183	2.8577	3.0042	3.1582	3.3201	3.4903

Sol: *Taking the origin at* $x = 1.15$ *i.e* $x_0 = 1.15$; $x = 1.17$; $h = 0.05$

$u = \dfrac{x - x_0}{h} = \dfrac{1.17 - 1.15}{0.05} = \dfrac{0.02}{0.05} = 0.4$

Construct the difference table is given as follows

x	$u=\frac{x-x_0}{h}$	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$
1.00	-3	2.7183				
			0.1394			
1.05	-2	2.8577		0.0071		
			0.1465		0.0004	
1.10	-1	3.0042		0.0075		0
			0.1540		0.0004	
1.15	0		F	0.0079		0
		3.1582				
			0.1619		0.0004	
1.20	1			0.0083		
		3.3201				
			0.1702			
1.25	2					
		3.4903				

$$The\ Gauss\ forward\ formula\ is\ \ y_x=y_0+u\Delta y_0+\frac{u(u-1)}{2!}\Delta^2 y_{-1}+\frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1}$$

$$+\frac{u(u-1)(u+1)(u-2)}{4!}\Delta^4 y_{-2}+\frac{u(u-1)(u+1)(u-2)(u+2)}{5!}\Delta^5 y_{-2}+\cdots$$

$$=3.1582+0.4(0.1619)+\frac{0.4(0.4-1)}{2}(0.0079)+\frac{0.4(0.4-1)(0.4+1)}{6}(0.0004)+0$$

$$=3.1582+0.06476-0.000948-0.0000224=3.2221\ approximately$$

5. *Use Gauss's backward formula to find* $\sqrt{12516}$; *given that* $\sqrt{12500}=111.803399$; $\sqrt{12510}=111.848111$; $\sqrt{12520}=111.892806$; $\sqrt{12530}=111.937483$

Sol: *Taking the origin at* $x=12520$ *i.e* $x_0=12520$; $x=12516$; $h=10$

$$u=\frac{x-x_0}{h}=\frac{12516-12520}{10}=\frac{-4}{10}=-0.4$$

Construct the difference table is given as follows

x	$u=\dfrac{x-x_0}{h}$	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$
12500	-2	111.803399			
			0.044712		
12510	-1	111.848111		-0.000017	
			0.044695		-0.000001
12520	0	111.892806		0.000018	
			0.044677		
12530	1	111.937483			

The Gauss backward formula is $y_x = y_0 + u\Delta y_{-1} + \dfrac{u(u+1)}{2!}\Delta^2 y_{-1} + \dfrac{u(u-1)(u+1)}{3!}\Delta^3 y_{-2} + \cdots$

$$y_{12516} = 111.892806 + (-0.4)(0.044695) + \frac{-0.4(-0.4+1)}{2}(-0.000018)$$
$$+\frac{-0.4(-0.4-1)(-0.4+1)}{6}(-0.000001)$$

$= 111.892806 - 0.017878 + 0.000002 - 0.000000056 = 111.874930$ *approximately*

6. *Using Striling's formula find $f(1.22)$ from the following data.*

x	1.0	1.1	1.2	1.3	1.4
y	0.841	0.891	0.932	0.963	0.985

Sol: *Taking the origin at* $x = 1.2$ *i.e* $x_0 = 1.2;\ \ x = 1.22;\ \ h = 0.1$

$$u = \frac{x-x_0}{h} = \frac{1.22-1.2}{0.1} = \frac{0.02}{0.1} = 0.2$$

Construct the difference table is given as follows

x	$u=\frac{x-x_0}{h}$	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$
1.0	-2	0.841				
			0.050			
1.1	-1	0.891		-0.009		
			0.041		-0.001	
1.2	0	0.932		-0.010		0.002
			0.031		0.001	
1.3	1			-0.009		
		0.963				
			0.022			
1.4	2					
		0.985				

$$\text{The Striling formula is } y_x = y_0 + \frac{u}{2}(\Delta y_0 + \Delta y_{-1}) + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$

$$+\frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

$$= 0.932 + \frac{0.2}{2}(0.031+0.041) + \frac{(0.2)^2}{2}(-0.010) + \frac{(0.2)((0.2)^2-1)}{6}\left[\frac{-0.001+0.001}{2}\right]$$

$$+\frac{(0.2)^2((0.2)^2-1)}{24}(0.002)$$

$$= 0.932 + 0.0072 - 0.002 + 0.00002 = 0.93718 \approx 0.937 \text{ approximately}$$

$$\therefore f(1.22) = 0.937$$

7. *Using Striling's formula find $f(1.91)$ from the following data.*

x	1.7	1.8	1.9	2.0	2.1	2.2
$y=e^x$	5.4739	6.0496	6.6859	7.3891	8.1662	9.0250

Sol: *Taking the origin at* $x = 1.9$ *i.e* $x_0 = 1.9$; $x = 1.91$; $h = 0.1$

$$u = \frac{x-x_0}{h} = \frac{1.91-1.9}{0.1} = \frac{0.01}{0.1} = 0.1$$

Construct the difference table is given as follows

x	$u=\dfrac{x-x_0}{h}$	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$	$\Delta^5 y_u$
1.7	-2	5.4739					
			0.5757				
1.8	-1	6.0486		0.0606			
			0.6363		0.0063		
1.9	0	$\boxed{6.6859}$		0.0669		0.0007	
			0.7032		0.0070		
2.0	1			0.0739		0.0008	0.0001
		7.3891					
			0.7771		0.0078		
2.1	2			0.0817			
		8.1662					
			0.8588				
2.2	3						
		9.0250					

$$The\ Striling\ \ formula\ is\ y_x = y_0 + \frac{u}{2}(\Delta y_0 + \Delta y_{-1}) + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$

$$+\frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

$$= 6.6859 + \frac{0.1}{2}(0.7032 + 0.6363) + \frac{(0.1)^2}{2}(0.669) + \frac{(0.1)((0.1)^2-1)}{6}\left[\frac{0.007 + 0.0063}{2}\right]$$

$$+\frac{(0.1)^2((0.1)^2-1)}{24}(0.0007)$$

$$= 6.6859 + 0.06975 + 0.0003345 - 0.000109725 - 0.0000002875 = 6.753099486$$

$$\approx 6.7531\ approximately$$

$$\therefore f(1.91) = 6.7531$$

8. *Using Striling's formula find f(25) from the following data.*

x	10	20	30	40
y	1.1	2	4.4	7.9

Sol: $Taking\ the\ origin\ at\ x = 20\ i.e\ x_0 = 20;\ \ x = 25;\ \ h = 10$

$$u = \frac{x - x_0}{h} = \frac{25 - 20}{10} = \frac{5}{10} = 0.5$$

Construct the difference table is given as follows

x	$u = \frac{x - x_0}{h}$	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$
10	-1	1.1			
			0.9		
20	0	2		1.5	
			2.4		-0.4
30	1	4.4		1.1	
			3.5		
40	2	7.9			

$$The\ Striling\ \ formula\ is\ y_x = y_0 + \frac{u}{2}(\Delta y_0 + \Delta y_{-1}) + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u(u^2 - 1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$

$$+ \frac{u^2(u^2 - 1)}{4!}\Delta^4 y_{-2} + \cdots$$

$$= 2 + \frac{0.5}{2}(0.9 + 2.4) + \frac{(0.5)^2}{2}(1.5) + \frac{(0.5)((0.5)^2 - 1)}{6}\left[\frac{-0.4 + 0}{2}\right]$$

$$= 2 + 0.825 + 0.1875 + 0.0125 = 3.025\ approximately$$

$$\therefore f(25) = 3.025$$

9. *Using Striling's formula find y_{35} from $y_{20} = 512;\ y_{30} = 439;\ y_{40} = 346;\ y_{50} = 243$*

Sol: $Taking\ the\ origin\ at\ x = 30\ i.e\ x_0 = 30;\ \ x = 35;\ \ h = 10$

$$u = \frac{x - x_0}{h} = \frac{35 - 30}{10} = \frac{5}{10} = 0.5$$

Construct the difference table is given as follows

x	$u=\frac{x-x_0}{h}$	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$
20	-1	512			
			-73		
30	0	439		-20	
			-93		10
40	1	346		-10	
			-103		
50	2				
		243			

$$The\ Striling\ \ formula\ is\ y_x = y_0 + \frac{u}{2}(\Delta y_0 + \Delta y_{-1}) + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$

$$+\frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

$$= 439 + \frac{0.5}{2}(-93-73) + \frac{(0.5)^2}{2}(-20) = 439 - 41.5 - 2.5 = 395$$

$\therefore \boldsymbol{y_{35}} = 395$

$10. Using\ Bessel's\ formula\ find\ y_{25}\ from\ y_{20} = 2854;\ y_{24} = 3162;\ y_{28} = 3544;\ y_{32} = 3992$

Sol: $Taking\ the\ origin\ at\ x = 24\ i.e\ x_0 = 24;\ \ x = 25;\ \ h = 4$

$$u = \frac{x - x_0}{h} = \frac{25-24}{4} = \frac{1}{4} = 0.25$$

Construct the difference table is given as follows

x	$u = \frac{x - x_0}{h}$	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$
20	-1	2854			
			308		
24	0	3162		74	
			382		-8
28	1	3544		66	
			448		
32	2	3992			

The Bessel's formula is

$$y_x = \frac{y_0 + y_1}{2} + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{u(u-1)\left(u - \frac{1}{2}\right)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{u(u+1)(u-1)(u-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \cdots$$

$$= \frac{1}{2}[3162 + 3544] + \left(\frac{1}{4} - \frac{1}{2}\right)382 + \frac{1}{4}\left(\frac{1}{4} - 1\right)\left(\frac{74 + 66}{4}\right) + \left(\frac{1}{4} - \frac{1}{2}\right)\frac{1}{4}\left(\frac{1}{4} - 1\right)\left(-\frac{8}{6}\right)$$

$$= 3353 - 95.5 - 6.5625 - 0.0625 = 3250.875$$

$\therefore \boldsymbol{y_{25}} = 3250.875$

11. ***Using Gauss backward's formula find $f(1976)$ from the following data.***

Year: x	1940	1950	1960	1970	1980	1990	Ans: f(1976)=34.4lakhs
Sales in lakhs: y	17	20	27	32	36	38	

12. ***Use stirling's formula to find polynomial from the data***

x	1	2	3	4	5	Ans: $f(x) = 1 + 2(x-3)^2 + \frac{2}{3}(x - 3)^2[(x-3)^2 - 1]$
y	1	-1	1	-1	1	

Interpolation with unequal intervals

Explain divided differences and write its properties

Divided differences: -The differences defined by taking into consideration the change in the values of arguments is called as divided differences

In this explicit function $y = f(x)$ where $x \in [x_0\ x_n]$, consider the argument $x_0, x_1, x_2, \dots, x_n$ are unequal differences i.e.$x_1 - x_0, x_2 - x_1, \dots, x_n - x_{n-1}$ need not be equal. Now the corresponding entries are $f(x_0), f(x_1), f(x_2), \dots, f(x_n)$.

$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$ is called the first divided differences and it is denoted by $f(x_0, x_1)\ or \underset{x_1}{\Delta} f(x_0)$

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \underset{x_1}{\Delta} f(x_0)$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \underset{x_2}{\Delta} f(x_1)$$

$\vdots$

$f(x_{n-1}, x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} = \underset{x_n}{\Delta} f(x_{n-1})$ are called first divided differences.

The differences of first divided differences are called second divided differences

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} = \underset{x_1, x_2}{\Delta^2} f(x_0)$$

$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1} = \underset{x_2, x_3}{\Delta^2} f(x_1)$$

$\vdots$

$f(x_0, x_1 \dots x_n) = \frac{f(x_1, x_2 \dots x_n) - f(x_0, x_1 \dots x_{n-1})}{x_n - x_0} = \underset{x_1, x_2, \dots x_n}{\Delta^2} f(x_0)$ This is n^{th} divided differences

Properties: -1. The divided differences are symmetric i.e. $f(x_0, x_1) = f(x_1, x_0)$

$$f(x_0, x_1, x_2) = f(x_1, x_0, x_2) = f(x_1, x_2, x_0) = \cdots$$

2. Divided differences operator is linear i.e. $\Delta [k_1 f_1(x) + k_2 f_2(x)] = k_1 \Delta f_1(x) + k_2 \Delta f_2(x)$

3. The n^{th} order divided differences of a polynomial of degree ‘n’ are constants.

Newton’s divided difference formula: - If $y = f(x)$ is a function, and $f(x_0), f(x_1), f(x_2), \dots, f(x_n)$ are the entries corresponding to the arguments $x_0, x_1, x_2, \dots, x_n$ are not equally spaced, then $f(x) = f(x_0) +$

$(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)+\cdots+(x-x_0)(x-x_1)\ldots(x-x_{n-1})f(x_0,x_1,\ldots x_n)$

Proof) Let $f(x_0),f(x_1),f(x_2),\ldots f(x_n)$ be the entries corresponding to the arguments $x_0,x_1,x_2,\ldots,x_n$ are not equally spaced. We know that the first divided differences of $f(x)$ is

$$f(x,x_0)=\frac{f(x)-f(x_0)}{x-x_0}\Rightarrow f(x)=f(x_0)+(x-x_0)f(x,x_0)\rightarrow(1)$$

Next $f(x,x_0,x_1)=\frac{f(x,x_0)-f(x_0,x_1)}{x-x_1}\Rightarrow f(x,x_0)=f(x_0,x_1)+(x-x_1)f(x,x_0,x_1)\rightarrow(2)$

Now substitute (2) in (1) we get

$$f(x)=f(x_0)+(x-x_0)[f(x_0,x_1)+(x-x_1)f(x,x_0,x_1)]$$

$$=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x,x_0,x_1)\rightarrow(3)$$

Again $f(x,x_0,x_1,x_2)=\frac{f(x,x_0,x_1)-f(x_0,x_1,x_2)}{x-x_2}\Rightarrow f(x,x_0,x_1)=f(x_0,x_1,x_2)+(x-x_2)f(x,x_0,x_1,x_2)\rightarrow(4)$

Now substitute (4) in (3) we get

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+(x-x_0)(x-x_1)(x-x_2)f(x,x_0,x_1,x_2)$$

Proceeding in the same way we get $f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)+\cdots+(x-x_0)(x-x_1)\ldots(x-x_{n-1})f(x_0,x_1,\ldots x_n)$

Lagrange's interpolation formula: -- If $y=f(x)$ is a function, and $f(x_0),f(x_1),f(x_2),\ldots,f(x_n)$ are the entries corresponding to the arguments $x_0,x_1,x_2,\ldots,x_n$ are not equally spaced, then

$$f(x)=\frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}f(x_0)+\frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}f(x_1)+\cdots+\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}f(x_n)$$

Proof) Let $f(x_0),f(x_1),f(x_2),\ldots f(x_n)$ be the entries corresponding to the arguments $x_0,x_1,x_2,\ldots,x_n$ are not equally spaced. Define a polynomial $f(x)=(x-x_1)(x-x_2)\ldots(x-x_n)A_0+(x-x_0)(x-x_2)\ldots(x-x_n)A_1+(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)A_2+\cdots+(x-x_0)(x-x_1)\ldots(x-x_{n-1})A_n\rightarrow(1)$ where $A_0,A_1,A_2,\ldots A_n$ are constants.

Put $x=x_0$ in (1) we get $f(x_0)=(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)A_0\Rightarrow A_0=\frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}$

Put $x=x_1$ in (1) we get $f(x_1)=(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)A_1\Rightarrow A_1=\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}$

Put $x=x_2$ in (1) we get $f(x_2)=(x_2-x_0)(x_2-x_1)\ldots(x_2-x_n)A_2\Rightarrow A_2=\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)\ldots(x_2-x_n)}$

Proceeding in this way we get$\Rightarrow A_n=\frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}$

Substitute all these values in (1) we get

$$f(x)=\frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}f(x_0)+\frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}f(x_1)+\cdots+\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}f(x_n)$$

Inverse interpolation: -Let $y=f(x)$ be an unknown function of independent variable 'x' .Suppose (x_0,y_0), $(x_1,y_1),(x_2,y_2),\ldots,(x_i,y_i),\ldots,(x_n,y_n)$ are given set of tabulated values satisfying $y=f(x)$

The process of finding the values of the argument 'x' corresponding to a given values of the function y lying between two tabulated values y_i $where\ i=1,2,3,\ldots n$ with the help of the given set of observations x_i $where\ i=1,2,3,\ldots n$ is known as "inverse interpolation".

1. *Show that 1 st divided difference is symmatric*

Sol: $f(x_0,x_1)=\dfrac{f(x_1)-f(x_0)}{x_1-x_0}=\dfrac{f(x_0)-f(x_1)}{x_0-x_1}=f(x_1,x_0)$

$\therefore f(x_0,x_1)=f(x_1,x_0)$ *so 1st divided difference is symmetric*

2. $f(x)=\dfrac{1}{x}$ *then find* $\Delta f(a,b)$*and* $\Delta f(a,b,c)$

Sol: $f(x)=\dfrac{1}{x}$ $\quad (i)\ \Delta f(a,b)=\dfrac{f(b)-f(a)}{b-a}=\dfrac{\frac{1}{b}-\frac{1}{a}}{b-a}=\dfrac{\frac{a-b}{ab}}{b-a}=-\dfrac{(b-a)}{(ab)(b-a)}=-\dfrac{1}{ab}$

$$(ii)\ \Delta f(a,b,c)=\frac{\Delta f(b,c)-\Delta f(a,b)}{c-a}=\frac{\frac{-1}{bc}-\left(\frac{-1}{ab}\right)}{c-a}=\frac{\frac{-a+c}{abc}}{c-a}=\frac{1}{abc}$$

3. $f(x)=\dfrac{1}{x^2}$ *then find* $\Delta f(a,b)$, $\Delta f(a,b,c)$ *and* $\Delta f(a,b,c,d)$

Sol: $f(x)=\dfrac{1}{x^2}$

$$(i)\ \Delta f(a,b)=\frac{f(b)-f(a)}{b-a}=\frac{\frac{1}{b^2}-\frac{1}{a^2}}{b-a}=\frac{\frac{a^2-b^2}{a^2b^2}}{b-a}=\frac{-(b-a)(a+b)}{(b-a)(a^2b^2)}=-\frac{a+b}{a^2b^2}$$

$$(ii)\ \Delta f(a,b,c)=\frac{\Delta f(b,c)-\Delta f(a,b)}{c-a}=\frac{-\frac{b+c}{b^2c^2}-\left(-\frac{a+b}{a^2b^2}\right)}{c-a}=\frac{\frac{-(b+c)a^2+(a+b)c^2}{a^2b^2c^2}}{c-a}$$

$$=\frac{b(c^2-a^2)+ac(c-a)}{(c-a)a^2b^2c^2}=\frac{b(c+a)}{a^2b^2c^2}+\frac{ac}{a^2b^2c^2}=\frac{ab+bc+ca}{a^2b^2c^2}$$

$$(ii)\ \Delta f(a,b,c,d)=\frac{\Delta f(b,c,d)-\Delta f(a,b,c)}{d-a}=\frac{\frac{bc+cd+bd}{b^2c^2d^2}-\left(\frac{ab+bc+ca}{a^2b^2c^2}\right)}{d-a}$$

$$= \frac{\frac{(bc+cd+db)a^2 - (ab+bc+ca)d^2}{a^2b^2c^2d^2}}{d-a} = \frac{(bc+cd+db)a^2 - (ab+bc+ca)d^2}{(d-a)a^2b^2c^2d^2}$$

$$= \frac{bc(a^2-d^2) + abd(a-d) + acd(a-d)}{(d-a)a^2b^2c^2d^2} = -\frac{abc+bcd+cda+dab}{a^2b^2c^2d^2}$$

4. Evluate $\underset{y}{\Delta}\, x^2$

Sol: We know that $\underset{y}{\Delta}\, f(x) = \frac{f(y)-f(x)}{y-x} = \frac{y^2-x^2}{y-x} = y+x$

5. If $f(x) = x$ then find $\Delta f(\alpha,\beta)$

Sol: $f(x) = x$ $\quad \Delta f(\alpha,\beta) = \frac{f(\beta)-f(\alpha)}{\beta-\alpha} = \frac{\beta-\alpha}{\beta-\alpha} = 1$

6. $f(x) = \frac{1}{x^2}$ then find $\Delta f(2,3)$

Sol: $f(x) = \frac{1}{x^2}$ $\quad (i)\ \Delta f(2,3) = \frac{f(3)-f(2)}{3-2} = \frac{\frac{1}{3^2}-\frac{1}{2^2}}{1} = \frac{1}{9}-\frac{1}{4} = \frac{4-9}{36} = -\frac{5}{36}$

7. Find the 3 rd divided difference of $f(x) = x^3 + x + 2$ for the arguments 1, 3, 6, 11

Sol: Construct the divided difference table as follows

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	4			
		$\frac{32-4}{3-1} = 14$		
			$\frac{64-14}{6-1} = 10$	
3	32			$\frac{20-10}{11-1} = 1$
		$\frac{224-32}{6-3} = 64$		
			$\frac{224-64}{11-3} = 20$	
6	224	$\frac{1344-224}{11-6} = 224$		
11	1344			

Hence the 3rd divided difference is 1

$8.$ *Find the polynomial from given data*

x	0	1	2	5
y	2	3	12	147

Sol: *Construct the divided difference table as follows*

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	2			
		$\frac{3-2}{1-0}=1$		
			$\frac{9-1}{2-0}=4$	
1	3			$\frac{9-4}{5-0}=1$
		$\frac{12-3}{2-1}=9$		
			$\frac{45-9}{5-1}=9$	
2	12	$\frac{147-12}{5-2}=45$		
5	147			

The Newton's divided difference formula is $f(x) = f(x_0) + (x-x_0)f(x_0,x_1)$

$+(x-x_0)(x-x_1)f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3) + \cdots$

$+(x-x_0)(x-x_1)\ldots(x-x_{n-1})f(x_0,x_1,\ldots x_n)$

$\Longrightarrow f(x) = f(0) + (x-0)f(x_0,x_1) + (x-0)(x-1)f(x_0,x_1,x_2) + (x-0)(x-1)(x-2)f(x_0,x_1,x_2,x_3)$

$= 2 + x(1) + x(x-1)(4) + x(x-1)(x-2)(1) = 2 + x + 4x^2 - 4x + x^3 - 3x^2 + 2x$

$\Longrightarrow f(x) = x^3 + x^2 - x + 2$

$9.$ *Using Newton's divided difference formula obtain value of y when $x = 2$ for the set of tabulated points* $(1,-3),(3,9),(4,30)$ *and* $(6,132)$

Sol: *Construct the divided difference table as follows*

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	-3			
		$\frac{9-(-3)}{3-1}=6$		
			$\frac{21-6}{4-1}=5$	
3	9			$\frac{10-5}{6-1}=1$
		$\frac{30-9}{4-3}=21$		
			$\frac{51-21}{6-3}=10$	
4	30			
		$\frac{132-30}{6-4}=51$		
6	132			

The Newton's divided difference formula is $f(x)= \ f(\ x_0)+(x-x_0)f(x_0,x_1)$

$+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)+\cdots$

$+(x-x_0)(x-x_1)\dots(x-x_{n-1})f(x_0,x_1,\dots x_n)$

$\Longrightarrow f(x)= \ f(\ 1)+(x-1)f(x_0,x_1)+(x-1)(x-3)f(x_0,x_1,x_2)+(x-1)(x-3)(x-4)f(x_0,x_1,x_2,x_3)$

$f(2)=-3+(2-1)(6)+(2-1)(2-3)(5)+(2-1)(2-3)(2-4)(1)=-3+6-5+2=-1$

$\Longrightarrow f(2)=-1$

10. ***Using Newton's divided difference formula obtain value of*** $\boldsymbol{f(9)}$ ***for the set of***

x	5	7	11	13	17
y	150	392	1452	2366	5202

Sol: *Construct the divided difference table as follows*

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
5	150				
		$\frac{392-150}{7-5}=121$			
			$\frac{265-121}{11-5}=24$		
7	392			$\frac{32-24}{13-5}=1$	
		$\frac{1452-392}{11-7}=265$			
			$\frac{457-265}{13-7}=32$		
11	1452				$\frac{1-1}{17-5}=0$
				$\frac{42-32}{17-7}=1$	
		$\frac{2366-1452}{13-11}=457$			
			$\frac{709-457}{17-11}=42$		
13	2366				
		$\frac{5202-2366}{17-13}=709$			
17	5202				

The Newton's divided difference formula is $f(x) = f(x_0) + (x-x_0)f(x_0,x_1)$

$+(x-x_0)(x-x_1)f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3) + \cdots$

$+(x-x_0)(x-x_1)\ldots(x-x_{n-1})f(x_0,x_1,\ldots x_n)$

$\Longrightarrow f(x) = f(5) + (x-5)f(x_0,x_1) + (x-5)(x-7)f(x_0,x_1,x_2)$

$+(x-5)(x-7)(x-11)f(x_0,x_1,x_2,x_3) + (x-5)(x-7)(x-11)(x-13)f(x_0,x_1,x_2,x_3,x_4)$

$f(9) = 150 + +(9-5)f(x_0,x_1) + (9-5)(9-7)f(x_0,x_1,x_2)$

$+(9-5)(9-7)(9-11)f(x_0,x_1,x_2,x_3) + (9-5)(9-7)(9-11)(9-13)f(x_0,x_1,x_2,x_3,x_4)$

$= 150 + 4(121) + 4(2)(24) + 4(2)(-2)(1) + 4(2)(-2)(-4)(0) = 150 + 484 + 192 - 16 + 0 = 810$

$\therefore f(9) = 810$

11. *Using Newton's divided difference formula obtain $f(x)$ for the set of*

x	0	1	2	4	5	6
y	1	14	15	5	6	19

Sol: *Construct the divided difference table as follows*

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		$\frac{14-1}{1-0}=13$			
			$\frac{13-1}{2-0}=6$		
1	14			$\frac{6-(-2)}{4-0}=2$	
		$\frac{15-14}{2-1}=1$			
			$\frac{-5-1}{4-1}=-2$		
2	15				$\frac{1-2}{5-0}=-\frac{1}{5}$
				$\frac{2-(-2)}{5-1}=1$	
		$\frac{5-15}{4-2}=-5$			
4	5				
			$\frac{1-(-5)}{5-2}=2$		
		$\frac{6-5}{5-4}=1$			
5	6				

The Newton's divided difference formula is $f(x) = f(x_0) + (x-x_0)f(x_0,x_1)$

$+(x-x_0)(x-x_1)f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3) + \cdots$

$+(x-x_0)(x-x_1)\ldots(x-x_{n-1})f(x_0,x_1,\ldots x_n)$

$\Longrightarrow f(x) = f(0) + (x-0)f(x_0,x_1) + (x-0)(x-1)f(x_0,x_1,x_2)$

$+(x-0)(x-1)(x-2)f(x_0,x_1,x_2,x_3) + (x-0)(x-1)(x-2)(x-4)f(x_0,x_1,x_2,x_3,x_4)$

$$= 1 + x(13) + x(x-1)(6) + x(x-1)(x-2)(2) + x(x-1)(x-2)(x-4)\left(\frac{1}{5}\right)$$

$$= 1 + 13x + 6x^2 - 6x + 2(x^3 - 3x^2 + 2x) + \frac{1}{5}(x^3 - 3x^2 + 2x)(x-4)$$

$$= 1 + 13x + 6x^2 - 6x + 2x^3 - 6x^2 + 6x + \frac{1}{5}(x^4 - 3x^3 + 2x^2 - 4x^3 + 12x^2 - 8x)$$

$$= \frac{1}{5}(5 + 65x + 10x^3 + x^4 - 7x^3 + 14x^2 - 8x) = \frac{1}{5}(x^4 + 3x^3 + 14x^2 + 57x + 5)$$

$$\therefore f(x) = \frac{1}{5}(x^4 + 3x^3 + 14x^2 + 57x + 5)$$

12. Find the polynomial of (3 3), (2 12), (1 15), (−1 − 21) by useing Newton's divided formula

13. Use Lagranges interpolation formula to fit a polynomial to the data

x	-1	0	2	3
y	-8	3	1	12

Sol: *Here* $x_0 = -1, x_1 = 0, x_2 = 2, x_3 = 3$ *The Lagrange's interpolation formula is*

$$f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0) + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1) +$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f(x_2) + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} f(x_3)$$

$$= \frac{(x - 0)(x - 2)(x - 3)}{(-1 - 0)(-1 - 2)(-1 - 3)}(-8) + \frac{(x + 1)(x - 2)(x - 3)}{(0 + 1)(0 - 2)(0 - 3)}(3) +$$

$$+ \frac{(x + 1)(x - 0)(x - 3)}{(2 + 1)(2 - 0)(2 - 3)}(1) + \frac{(x + 1)(x - 0)(x - 2)}{(3 + 1)(3 - 0)(3 - 2)}(12)$$

$$= \frac{(x - 0)(x - 2)(x - 3)}{(-12)}(-8) + \frac{(x + 1)(x - 2)(x - 3)}{6}(3) +$$

$$+ \frac{(x + 1)(x - 0)(x - 3)}{(-6)}(1) + \frac{(x + 1)(x - 0)(x - 2)}{12}(12)$$

$$= \frac{2}{3}x(x^2 - 5x + 6) + \frac{1}{2}(x + 1)(x^2 - 5x + 6) - \frac{1}{6}x(x^2 - 2x - 3) + x(x^2 - x - 2)$$

$$= \frac{2}{3}(x^3 - 5x^2 + 6x) + \frac{1}{2}(x^3 + x^2 - 5x^2 - 5x + 6x + 6) - \frac{1}{6}(x^3 - 2x^2 - 3x) + (x^3 - x^2 - 2x)$$

$$= \frac{2}{3}(x^3 - 5x^2 + 6x) + \frac{1}{2}(x^3 - 4x^2 + x + 6) - \frac{1}{6}(x^3 - 2x^2 - 3x) + (x^3 - x^2 - 2x)$$

$$= x^3\left(\frac{2}{3} + \frac{1}{2} - \frac{1}{6} + 1\right) + x^2\left(\frac{-10}{3} - 2 + \frac{1}{3} - 1\right) + x\left(4 + \frac{1}{2} + \frac{1}{2} - 2\right) + 3$$

$$= 2x^3 - 6x^2 + 3x + 3$$

$$\therefore f(x) = 2x^3 - 6x^2 + 3x + 3$$

14. *Fit a cubic polynomial by useing Lagranges interpolation formula to the data*

x	-2	-1	2	3
y	-12	-8	3	5

Sol: *Here* $x_0 = -2, x_1 = -1, x_2 = 2, x_3 = 3$ *The Lagrange's interpolation formula is*

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) +$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$= \frac{(x+1)(x-2)(x-3)}{(-2+1)(-2-2)(-2-3)}(-12) + \frac{(x+2)(x-2)(x-3)}{(-1+2)(-1-2)(-1-3)}(-8) +$$

$$+ \frac{(x+2)(x+1)(x-3)}{(2+2)(2+1)(2-3)}(3) + \frac{(x+2)(x+1)(x-2)}{(3+2)(3+1)(3-2)}(5)$$

$$= \frac{(x+1)(x^2-5x+6)}{(-20)}(-12) + \frac{(x+2)(x^2-5x+6)}{12}(-8) +$$

$$+ \frac{(x-3)(x^2+3x+2)}{(-12)}(3) + \frac{(x-2)(x^2+3x+2)}{20}(5)$$

$$= \frac{3}{5}(x^3 + x^2 - 5x^2 - 5x + 6x + 6) - \frac{2}{3}(x^3 + 2x^2 - 5x^2 - 10x + 6x + 12)$$

$$- \frac{1}{4}(x^3 - 3x^2 + 3x^2 - 9x + 2x - 6) + \frac{1}{4}(x^3 - 2x^2 + 3x^2 - 6x + 2x - 4)$$

$$= \frac{3}{5}(x^3 - 4x^2 + x + 6) - \frac{2}{3}(x^3 - 3x^2 - 4x + 12) - \frac{1}{4}(x^3 - 7x - 6) + \frac{1}{4}(x^3 + x^2 - 4x - 4)$$

$$= x^3\left(\frac{3}{5} - \frac{2}{3} - \frac{1}{4} + \frac{1}{4}\right) + x^2\left(\frac{-12}{5} + 2 + \frac{1}{4}\right) + x\left(\frac{3}{5} + \frac{8}{3} + \frac{7}{4} - 1\right) + \frac{18}{5} - 8 + \frac{3}{2} - 1$$

$$= -\frac{1}{15}x^3 - \frac{3}{20}x^2 + \frac{241}{60}x - \frac{39}{10}$$

$$\therefore f(x) = -\frac{1}{15}x^3 - \frac{3}{20}x^2 + \frac{241}{60}x - \frac{39}{10}$$

15. *Use Lagranges interpolation formula to find* $y(10)$ ***to the data***

x	5	6	9	11
y	12	13	14	16

Sol: *Here* $x_0 = 5, x_1 = 6, x_2 = 9, x_3 = 11$ *The Lagrange's interpolation formula is*

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$= \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)}(13) +$$

$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)}(16)$$

$$f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$

$$= \frac{(4)(1)(-1)}{(-1)(-4)(-6)}(12) + \frac{(5)(1)(-1)}{(1)(-3)(-5)}(13) + \frac{(5)(4)(-1)}{(4)(3)(-2)}(14) + \frac{(5)(4)(1)}{(6)(5)(2)}(16)$$

$$= 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3} = 2 + \frac{38}{3} = 2 + 12.6 = 14.6$$

$\therefore y(10) = 14.6$

16. ***Use Lagranges interpolation formula to find*** $\boldsymbol{y(0)}$ ***to the data***

x	5	6	9	11
y	12	13	14	16

Sol: *Here* $x_0 = 5, x_1 = 6, x_2 = 9, x_3 = 11$ *The Lagrange's interpolation formula is*

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$= \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)}(16)$$

$$f(0)=\frac{(0-6)(0-9)(0-11)}{(5-6)(5-9)(5-11)}(12)+\frac{(0-5)(0-9)(0-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+\frac{(0-5)(0-6)(0-11)}{(9-5)(9-6)(9-11)}(14)+\frac{(0-5)(0-6)(0-9)}{(11-5)(11-6)(11-9)}(16)$$

$$=\frac{(-6)(-9)(-11)}{(-1)(-4)(-6)}(12)+\frac{(-5)(-9)(-11)}{(1)(-3)(-5)}(13)+\frac{(-5)(-6)(-11)}{(4)(3)(-2)}(14)+\frac{(-5)(-6)(-9)}{(6)(5)(2)}(16)$$

$$=297-429+\frac{385}{2}-72=297-429+192.5-72=-11.5$$

$$\therefore y(0)=-11.5$$

17. *For the given values of x and y find $\sin\left(\frac{\pi}{6}\right)$ useing Lagranges interpolation formula*

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$y=sinx$	0	0.70711	1

Sol: *Here* $x_0=0, x_1=\frac{\pi}{4}, x_2=\frac{\pi}{2}$ *The Lagrange's interpolation formula is*

$$f(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

$$=\frac{\left(x-\frac{\pi}{4}\right)\left(x-\frac{\pi}{2}\right)}{\left(0-\frac{\pi}{4}\right)\left(0-\frac{\pi}{2}\right)}(0)+\frac{(x-0)\left(x-\frac{\pi}{2}\right)}{\left(\frac{\pi}{4}-0\right)\left(\frac{\pi}{4}-\frac{\pi}{2}\right)}(0.70711)+\frac{(x-0)\left(x-\frac{\pi}{4}\right)}{\left(\frac{\pi}{2}-0\right)\left(\frac{\pi}{2}-\frac{\pi}{4}\right)}(1)$$

$$\sin\left(\frac{\pi}{6}\right)=0+\frac{\left(\frac{\pi}{6}-0\right)\left(\frac{\pi}{6}-\frac{\pi}{2}\right)}{\left(\frac{\pi}{4}-0\right)\left(\frac{\pi}{4}-\frac{\pi}{2}\right)}(0.70711)+\frac{\left(\frac{\pi}{6}-0\right)\left(\frac{\pi}{6}-\frac{\pi}{4}\right)}{\left(\frac{\pi}{2}-0\right)\left(\frac{\pi}{2}-\frac{\pi}{4}\right)}(1)$$

$$=\frac{\left(\frac{\pi}{6}\right)\left(-\frac{\pi}{3}\right)}{\left(\frac{\pi}{4}\right)\left(-\frac{\pi}{4}\right)}(0.70711)+\frac{\left(\frac{\pi}{6}\right)\left(-\frac{\pi}{12}\right)}{\left(\frac{\pi}{2}\right)\left(\frac{\pi}{4}\right)}(1)=\frac{16}{18}(0.70711)-\frac{8}{72}=\frac{8}{9}(0.70711)-\frac{1}{9}=\frac{4.65688}{9}$$

$$\therefore \sin\left(\frac{\pi}{6}\right)=0.51743$$

18. *Use Lagranges inverse interpolation formula to find the value of x when $y=15$*

from the the data

x	5	6	9	11
y	12	13	14	16

Sol: $Here\ y_0 = 12, y_1 = 13, y_2 = 14, y_3 = 16$

$The\ Lagrange's\ inverse\ interpolation\ formula\ is$

$$f^{-1}(y) = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}x_1$$

$$+\frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}x_3$$

$$= \frac{(y-13)(y-14)(y-16)}{(12-13)(12-14)(12-16)}(5) + \frac{(y-12)(y-14)(y-16)}{(13-12)(13-14)(13-16)}(6)$$

$$+\frac{(y-12)(y-13)(y-16)}{(14-12)(14-13)(14-16)}(9) + \frac{(y-12)(y-13)(y-14)}{(16-12)(16-13)(16-14)}(11)$$

$$f^{-1}(15) = \frac{(15-13)(15-14)(15-16)}{(12-13)(12-14)(12-16)}(5) + \frac{(15-12)(15-14)(15-16)}{(13-12)(13-14)(13-16)}(6)$$

$$+\frac{(15-12)(15-13)(15-16)}{(14-12)(14-13)(14-16)}(9) + \frac{(15-12)(15-13)(15-14)}{(16-12)(16-13)(16-14)}(11)$$

$$= \frac{(2)(1)(-1)}{(-1)(-2)(-4)}(5) + \frac{(3)(1)(-1)}{(1)(-1)(-3)}(6) + \frac{(3)(2)(-1)}{(2)(1)(-2)}(9) + \frac{(3)(2)(1)}{(4)(3)(2)}(11)$$

$$= \frac{5}{4} - 6 + \frac{27}{2} + \frac{11}{4} = 1.25 - 6 + 13.5 + 2.75 = 10.5$$

$$\therefore x = f^{-1}(15) = 10.5$$

19. *Use Lagranges interpolation formula to fit a polynomial to the data*

x	0	1	2	5	$Ans: f(x) = x^3 + x^2 - x + 2$
y	2	3	12	147	

20. *Use Lagranges interpolation formula to find $f(4)$ to the data*

x	0	1	2	5
y	2	5	7	8

$Ans: f(4) = 8.4$

Numerical differentiation

Numerical differentiation is the process of evaluating the derivative of a function $f(x)$ with the help of the given set of that function is known as numerical differentiation.

Note: -If the derivative at a point near the beginning of set of values given by a table is required the we use Newton forward formula, and if the same is required at a point near the end of the given tabular values, then we use Newton backward formula, and also if the same is required at a point near the middle of the given tabular values, then we use Bessel's and Sterling's formula

Derive $\left[\frac{dy}{dx}\right]_{x=x_0}$ *and* $\left[\frac{d^2y}{dx^2}\right]_{x=x_0}$ by using (1) Newton's forward and backward formulas

Proof) Newton's forward interpolation is given by

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \to (1) where\ u = \frac{x-x_0}{h} \Rightarrow du = \frac{1}{h}dx$$

Differentiating (1) w.r.to u, we get

$$\frac{dy}{du} = \left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \cdots\right] \to (2) \text{ now } \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \cdots\right] \to (3)$$

As $x = x_0 \Rightarrow u = 0$ therefore, putting $u = 0$ in above equation

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right] \to (4)$$

Differentiating equation (3) again w.r.t. x we get

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right] \to (5)$$

Newton's backward interpolation is given by

$$y = y_0 + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \cdots \to (1) where\ u = \frac{x-x_n}{h} \Rightarrow du = \frac{1}{h}dx$$

Differentiating (1) w.r.to x, we get

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n + \frac{4u^3+18u^2+22u+6}{4!}\nabla^4 y_n + \cdots\right] \to (2)$$

As $x = x_n \Rightarrow u = 0$ therefore, putting $u = 0$ in above equation

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right] \to (3)$$

Differentiating equation (3) again w.r.t. x we get

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right] \to (4)$$

Sterling's formula, $\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\frac{\Delta y_0+\Delta y_{-1}}{2} - \frac{1}{6}\left(\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2}\right) + \cdots\right] \to (5)$

And $\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2+} \dots\right] \to (6)$

Maximum and minimum values of a tabulated function: -If we differentiate the Newton's forward interpolation with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \cdots\right] \to (3)$$

For maximum/minimum, put $\frac{dy}{dx} = 0 \Longrightarrow \Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \cdots = 0$ solving this for u and substitute the values from difference table we get x as $x_0 + uh$ at which y is maximum or minimum.

Errors in numerical differentiation: - The numerical computation of derivatives involves two types of errors namely truncation errors and rounding errors.

Truncation error: - The truncation error is caused by replacing the tabulated function $y = f(x)$ by means of an interpolating polynomial$\varphi(x)$. This error can be estimated by the formula $f(x) - \varphi(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\pi(x)$,

$x_0 < \xi < x_n \ \ where\ \pi(x) = (x - x_0)(x - x_1) \dots .(x - x_n).$

We can calculate the truncation error in any numerical differentiation formula in the following way. Suppose that the tabulated function is such that its difference of a certain order is small and the tabulated function is well approximated by the polynomial.

In sterling's formula, we can write $\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{\Delta y_{-1}+\Delta y_0}{2h} +$

$T_1 \ where\ T_1\ is\ the\ truncating\ error\ given\ by\ T_1 \ = \frac{1}{6h}\left|\frac{\Delta^3 y_{-2}+\Delta^3 y_{-1}}{2}\right|$

Similarly, $\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\Delta^2 y_{-1} + T_2 \ \ where\ T_2 = \frac{1}{12h^2}|\Delta^4 y_{-2}|$

Rounding Error: - The rounding error is inversely proportional to h in case of first derivatives, inversely proportional to h^2 in case of second derivatives and so on Thus the rounding error increases as h decreases.

In Sterling's formula, $\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{\Delta y_{-1}+\Delta y_0}{2h} - \frac{\Delta^3 y_{-2}+\Delta^3 y_{-1}}{12h} = \frac{y_{-2}-8y_{-1}+8y_1-y_2}{12h}$ has the maximum rounding error $\frac{18\varepsilon}{12h} = \frac{3\varepsilon}{2h}$ and the formula $\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\Delta^2 y_{-1} + \cdots = \frac{y_{-1}-2y_0+y_1}{h^2}$ has the maximum rounding error $\frac{4\varepsilon}{h^2}$

1. Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ $at\ x = 1.5$ from the following table

x	1.5	2.0	2.5	3.0	3.5	4.0
y	3.375	7	13.625	24	38.875	59

Sol) we form the difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1.5	3.375				
		3.625			
2	7.000		3.000		
		6.625		0.75	
2.5	13.625		3.75		0
		10.375		0.75	
3	24.000		4.5		0
		14.875		0.75	
3.5	38.875		5.25		
		20.125			
4	59.000				

Here $x_0 = 1.5,\ h = 0.5$ By Newton's forward formula $\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\ \Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots]$

$\Rightarrow \left[\frac{dy}{dx}\right]_{x=1.5} = \frac{1}{0.5}\left[3.625 - \frac{1}{2}(3) + \frac{1}{3}\ (0.75) - \frac{1}{4}(0)\right] = 2[2.375] = 4.75$

And $\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right] \Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=1.5} = \frac{1}{(0.5)^2}[3 - 0.75] = 4[2.25] = 9$

2. Find $\frac{dy}{dx}$, $at\ x = 1,3,6$ from the following table

x	0	1	2	3	4	5	6
y	6.9897	7.4036	7.7815	8.1291	8.4510	8.7506	9.0309

Sol) we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0	6.9897						
		0.4139					
1	7.4036		-0.0360				
		0.3779		0.0057			
2	7.7815		-0.0303		-0.0011		
		0.3476		0.0046		-0.0001	
3	8.1291		-0.0257		-0.0012		0.0009
		0.3219		0.0034		-0.0008	
4	8.4510		-0.0223		-0.0004		
		0.2996		0.0030			
5	8.7506		-0.0193				
		0.2803					
6	9.0309						

(i) Here $x_0 = 1,\ h = 1$ By Newton's forward formula $\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\ \Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$

$\Rightarrow \left[\frac{dy}{dx}\right]_{x=1} = \frac{1}{1}\left[0.3779 - \frac{1}{2}(-0.0303) + \frac{1}{3}\ (0.0046) - \frac{1}{4}(-0.0012) + \frac{1}{5}(-0.0008)\right] = 0.395043$

(ii) $x_0 = 3,\ h = 1$ Sterling's formula, $\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \cdots\right]$

$\Rightarrow \left[\frac{dy}{dx}\right]_{x=3} = \frac{1}{1}\left[\frac{(0.3476+0.3219)}{2} - \frac{1}{6}\left(\frac{0.0046+0.0034}{2}\right) + \cdots\right] = 0.33475 - 0.00067 = 0.33409$

(iii) $x_0 = 6,\ h = 1$ By Newton's backward formula $\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\ \nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots]$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{x=6} = \frac{1}{1}\left[0.2803 + \frac{1}{2}(-0.0193) + \frac{1}{3}\ (0.003) + \frac{1}{4}(-0.0004) + \cdots\right] = 0.27155$$

3. Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ $at\ x = 1$ from the following table

x	0	1	2	3	4
y	6.9897	7.4036	7.7815	8.1291	8.4510

Sol) we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	6.9897				
		0.4139			
1	7.4036		-0.0360		
		0.3779		0.0057	
2	7.7815		-0.0303		-0.0011
		0.3476		0.0046	
3	8.1291		-0.0257		
		0.3219			
4	8.4510				

(i) Here $x_0 = 1,\ h = 1$ By Newton's forward formula $\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\ \Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$

$\Rightarrow \left[\frac{dy}{dx}\right]_{x=1} = \frac{1}{1}\left[0.3779 - \frac{1}{2}(-0.0303) + \frac{1}{3}\ (0.0046)\right] = 0.3946$

And $\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right] \Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=1} = \frac{1}{(1)^2}[-0.0303 - 0.0046] = -0.0349$

4. Find the first and second derivatives of $\sqrt{x}$ $at\ x = 15$ from the following table

x	15	17	19	21	23	25
y	3.873	4.123	4.359	4.583	4.796	5.000

Sol) we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
15	3.873					
		0.250				
17	4.123		-0.014			
		0.236		0.002		
19	4.359		-0.012		-0.001	
		0.224		0.001		0.002
21	4.583		-0.011		0.001	
		0.213		0.002		
23	4.796		-0.009			
		0.204				
25	5.000					

(i) Here $x_0 = 15,\ h = 2$ By Newton's forward formula $\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\ \Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$

$\Rightarrow \left[\frac{dy}{dx}\right]_{x=15} = \frac{1}{2}\left[0.25 - \frac{1}{2}(-0.014) + \frac{1}{3}\ (0.002) - \frac{-0.001}{4} + \frac{0.002}{5}\right] = 0.129175$

And $\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right] \Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=15} = \frac{1}{(2)^2}[-0.014 - 0.002] = -0.004$

5. Find $\frac{d}{dx}j_0,\ at\ x = 0.1$ from the following table

x	0	0.1	0.2	0.3	0.4
y	1	0.9975	0.99	0.9776	0.9604

Sol) we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		-0.0025			
0.1	0.9975		-0.0050		
		-0.0075		0.0001	
0.2	0.99		-0.0049		0
		-0.0124		0.0001	
0.3	0.9776		-0.0048		
		-0.0172			
0.4	0.9604				

(i) Here $x_0 = 0.1,\ h = 0.1$ Newton's forward formula $\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\ \Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$

$\Rightarrow \left[\frac{dy}{dx}\right]_{x=0.1} = \frac{1}{0.1}\left[-0.0075 - \frac{1}{2}(-0.0049) + \frac{1}{3}\ (0.0001)\right] = -0.05017$

6. Find the first and second derivatives of y at $x = 0$ from the following table

x	0	2	4	6	8	10
y	0	12	248	1284	4080	9980

Sol) we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	0					
		12				
2	12		224			
		236		516		
4	248		800		384	
		1036		960		0
6	1284		1760		384	
		2796		1344		
8	4080		3104			
		5900				
10	9980					

(i) Here $x_0 = 0,\ h = 2$ By Newton's forward formula $\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$

$\Rightarrow \left[\frac{dy}{dx}\right]_{x=0} = \frac{1}{2}\left[12 - \frac{1}{2}(224) + \frac{1}{3}(576) - \frac{384}{4} + \frac{0}{5}\right] = -2$

And $\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 \dots\right] \Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=0} = \frac{1}{(2)^2}\left[224 - 576 + \frac{11}{12}(384)\right] = 0$

7. Find the first and second derivatives of y at $x = 0$ from the following table

x	0	1	2	3	4	5
y	4	8	15	7	6	2

Sol) we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	4					
		4				
1	8		3			
		7		-18		
2	15		-15		40	
		-8		22		-72
3	7		7		-32	
		-1		-10		
4	6		-3			
		-4				
5	2					

(i) Here $x_0 = 0,\ h = 1$ By Newton's forward formula $\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\ \Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$

$\Rightarrow \left[\frac{dy}{dx}\right]_{x=0} = \frac{1}{1}\left[4 - \frac{1}{2}(3) + \frac{1}{3}\ (-18) - \frac{40}{4} + \frac{-72}{5}\right] = -27.9$

And $\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 \ldots\right] \Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=0} = \frac{1}{(1)^2}\left[3 + 18 + \frac{11}{12}(40) - \frac{5}{6}(-72)\right]$

$= 117.67$

8. Find the first and second derivatives of $y\ at\ x = 1$ from the following table

x	1	2	3	4	5	6
y	1	8	27	64	125	216

Sol) we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1	1					
		7				
2	8		12			
		19		6		
3	27		18		0	
		37		6		0
4	64		24		0	
		61		6		
5	125		30			
		91				
6	216					

(i) Here $x_0 = 1,\ h = 1$ By Newton's forward formula $\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\ \Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$

$\Rightarrow \left[\frac{dy}{dx}\right]_{x=1} = \frac{1}{1}\left[7 - \frac{1}{2}(12) + \frac{1}{3}\ (6) - \frac{1}{4}(0)\right] = 3$

And $\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 \ldots\right] \Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=0} = \frac{1}{(1)^2}\left[12 - 6 + \frac{11}{12}(0)\right] = 6$

9. Find the derivative of f(x) at x = 0.4 from the following table.

x	0	0.1	0.2	0.3	0.4
y	1	0.9975	0.99	0.9776	0.9604

Sol) we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.1	1.10510			
		0.1163		
0.2	1.22140		0.01216	
		0.12846		0.00134
0.3	1.34986		0.0135	
		0.14196		
0.4	1.49182			

(i) $x_0 = 0.4,\ h = 0.1$ By Newton's backward formula $\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{x=0.4} = \frac{1}{0.1}\left[0.14196 + \frac{1}{2}(0.0135) + \frac{1}{3}(0.00134)\right] = 0.27155$$

10. Find the first derivative of y at $x = 1.6$ from the following table

x	1	1.2	1.4	1.6	1.8	2
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891

Sol) we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1	2.7183					
		0.6018				
1.2	3.3201		0.1333			
		0.7351		0.0294		
1.4	4.0552		0.1627		0.0067	
		0.8978		0.0361		0.0013
1.6	4.9530		0.1988		0.0080	
		1.0966		0.0441		
1.8	6.0496		0.2429			
		1.3395				
2	7.3891					

(i) Here $x_0 = 1.6,\ h = 0.2$ Sterling's formula, $\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \cdots\right]$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{x=1.6} = \frac{1}{0.2}\left[\frac{0.8978+1.0966}{2} - \frac{1}{6}\left(\frac{0.0361+0.0441}{2}\right)\right] = 5[0.9972 - 0.00668] = 4.9526$$

11. Find the first and second derivatives of y at $x = 1.2$ and 1.6 from the following table

x	1	1.2	1.4	1.6	1.8	2	2.2
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

Sol) we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1	2.7183						
		0.6018					
1.2	3.3201		0.1333				
		0.7351		0.0294			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	
1.6	4.9530		0.1988		0.0080		0.0001
		1.0966		0.0441		0.0014	
1.8	6.0496		0.2429		0.0094		
		1.3395		0.0535			
2	7.3891		0.2964				
		1.6359					
2.2	9.0250						

(i) $x_0 = 1.2,\ h = 0.2$ By Newton's forward formula $\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\ \Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$

$\Rightarrow \left[\frac{dy}{dx}\right]_{x=1.2} = \frac{1}{0.2}\left[0.7351 - \frac{1}{2}(0.1627) + \frac{1}{3}\ (0.0361) - \frac{1}{4}(0.008) + \frac{1}{5}(0.0014)\right] = 3.3203$

And $\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 \ldots\right] \Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=1.2} = \frac{1}{(0.2)^2}\left[0.1627 - 0.0361 + \frac{11}{12}(0.008) - \frac{5}{6}(0.0014)\right] = 3.3192$

(ii) Here $x_0 = 1.6,\ h = 0.2$ Sterling's formula, $\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \cdots\right]$

$\Rightarrow \left[\frac{dy}{dx}\right]_{x=1.6} = \frac{1}{0.2}\left[\frac{0.8978+1.0966}{2} - \frac{1}{6}\left(\frac{0.0361+0.0441}{2}\right)\right] = 5[0.9972 - 0.00668] = 4.9526$

And $\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2+} \ldots\right] \Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=1.6} = \frac{1}{(0.2)^2}\left[0.1988 - \frac{1}{12}(0.008)\right] = 4.9533$

12. Find the first and second derivatives of $y\ at\ x = 1.05$ from the following table

x	1	1.05	1.1	1.15	1.2	1.25	1.3
y	1	1.025	1.049	1.072	1.095	1.118	1.14

Sol) we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1	1						
		0.025					
1.05	1.025		-0.001				
		0.024		0			
1.1	1.049		-0.001		0.001		
		0.023		0.001		-0.002	
1.15	1.072		0		-0.001		0.002
		0.023		0		0	
1.2	1.095		0		-0.001		
		0.023		-0.001			
1.25	1.118		-0.001				
		0.022					
1.3	1.14						

(i) $x_0 = 1.05,\ h = 0.05$ By Newton's forward formula $\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\ \Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$

$\Rightarrow \left[\frac{dy}{dx}\right]_{x=1.05} = \frac{1}{0.05}\left[0.024 - \frac{1}{2}(-0.001) + \frac{1}{3}\ (0.001) - \frac{1}{4}(-0.001) + \frac{1}{5}(0)\right] = 0.5017$

And $\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 \ldots\right] \Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=1.05} = \frac{1}{(0.05)^2}\left[-0.001 - 0.001 + \frac{11}{12}(-0.001) - \frac{5}{6}(0)\right] = -1.1667$

13. Find the first and second derivatives of y at $x = 1$ from the following table

x	-2	-1	0	1	2	3	4
y	104	17	0	-1	8	69	272

Sol) we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
-2	104						
		-87					
-1	17		70				
		-17		-54			
0	0		16		48		
		-1		-6		0	
1	-1		10		48		0
		9		42		0	
2	8		52		48		
		61		90			
3	69		142				
		203					
4	272						

(ii) Here $x_0 = 1,\ h = 1$ Sterling's formula, $\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\frac{\Delta y_0+\Delta y_{-1}}{2} - \frac{1}{6}\left(\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2}\right) + \cdots\right]$

$\Rightarrow \left[\frac{dy}{dx}\right]_{x=1} = \frac{1}{1}\left[\frac{-1+9}{2} - \frac{1}{6}\left(\frac{-6+42}{2}\right)\right] = 1[4-3] = 1$

And $\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2+} \ldots\right] \Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=1} = \frac{1}{(1)^2}\left[10 - \frac{1}{12}(48)\right] = 10 - 4 = 6$

14. Find the first derivative of y at $x = 0.6$ by Newton's formula from the following table

x	0.2	0.4	0.6	0.8	1	1.2
y	0.122	0.493	1.123	2.022	3.20	4.666

Sol) we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.2	0.122					
		0.371				
0.4	0.493		0.259			
		0.63		0.01		
0.6	1.123		0.269		0	
		0.899		0.01		-0.001
0.8	2.022		0.279		-0.001	
		1.178		0.009		
1	3.20		0.288			
		1.466				
1.2	4.666					

(i) $x_0 = 0.6,\ h = 0.1$ By Newton's forward formula $\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$

$\Rightarrow \left[\frac{dy}{dx}\right]_{x=0.6} = \frac{1}{0.1}\left[0.899 - \frac{1}{2}(0.279) + \frac{1}{3}\ (0.009)\right] = 7.625$

15. Find the first second derivative of $y\ at\ x = 1.1$ from the following table

x	1	1.2	1.4	1.6	1.8	2
y	0	0.128	0.544	1.296	2.432	4

Sol) we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1	0					
		0.128				
1.2	0.128		0.288			
		0.416		0.048		
1.4	0.544		0.336		0	
		0.752		0.048		0
1.6	1.296		0.384		0	
		1.136		0.048		
1.8	2.432		0.432			
		1.568				
2	4					

Here we have to find the derivative at x =1.1 which lies between given arguments 1 and 1.2. So apply Newton's forward formula, we have $u = \frac{x-x_0}{h} = \frac{x-1}{0.2} = 5(x-1) \Rightarrow du = 5dx\ and\ u = 5(1.1-1) = 0.5$

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \rightarrow (1)$$

Differentiating (1) w.r.to u, we get

$\frac{dy}{du} = \left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \cdots\right] \rightarrow (2)$ now $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$$\frac{dy}{dx} = 5\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \cdots\right]$$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{x=1.1} = 5\left[0.128 + \frac{2(0.5)-1}{2!}(0.288) + \frac{3(0.5)^2-6(0.5)+2}{3!}(0.048)\right] = 5[0.128 - 0.002] = 0.63$$

And $\left[\frac{d^2y}{dx^2}\right]_{x=1.1} = 25[\Delta^2 y_0 + (u-1)\Delta^3 y_0] = 25[0.288 + (0.5-1)(0.048)] = 25[0.128 - 0.024] = 6.6$

16. Find the first and second derivatives of $y\ at\ x = 1$ from the following table

x	1	2	3	4	5	6
y	1	8	27	64	125	216

Sol) we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1	1					
		7				
2	8		12			
		19		6		
3	27		18		0	
		37		6		0
4	64		24		0	
		61		6		
5	125		30			
		91				
6	216					

Here we have to find the derivative at x =1.5 which lies between given arguments 1 and 2. So apply Newton's forward formula, we have $u = \frac{x-x_0}{h} = \frac{x-1.5}{1} = (x-1.5) \Rightarrow du = dx \text{ and } u = (1.5-1) = 0.5$

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \to (1)$$

Differentiating (1) w.r.to u, we get

$\frac{dy}{du} = \left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \cdots\right] \to (2)$ now $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

$$\frac{dy}{dx} = \left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \cdots\right]$$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{x=1.5} = \left[7 + \frac{2(0.5)-1}{2!}(12) + \frac{3(0.5)^2-6(0.5)+2}{3!}(6)\right] = [7-0.25] = 6.75$$

And $\left[\frac{d^2y}{dx^2}\right]_{x=1.1} = [\Delta^2 y_0 + (u-1)\Delta^3 y_0] = [12 + (0.5-1)(6)] = [12-3] = 9$

17. Find the derivative of f(x) at x = 1.22 from the following table.

x	1	1.1	1.2	1.3	1.4
y	0.84147	1.89121	0.93204	0.96356	0.98545

Sol) we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	0.84147				
		1.05			
1.1	1.89121		-2.009		
		-0.959		3.000	
1.2	0.93204		0.991		-4.001
		0.032		-1.001	
1.3	0.96356		-0.01		
		0.022			
1.4	0.98545				

Here $x_0 = 1.2,\ h = 0.1$ As the derivative is required near the middle of the table, we use Sterling's formula

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \cdots\right] = \frac{1}{0.1}\left[\frac{-0.959 + 0.032}{2} - \frac{1}{6}\left(\frac{3 - 1.001}{2}\right)\right] = -6.306$$

18. Find the derivative of f(x) at x = 4, 5 from the following table.

x	1	2	4	8	10
y	0	1	5	21	27

Sol) Since the vales of x are not equally spaced, we will use Newton's divided difference formula

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	0				
		$\frac{1-0}{2-1} = 1$			
2	1		$\frac{2-1}{4-1} = \frac{1}{3}$		
		$\frac{5-1}{4-2} = 2$		$\frac{\frac{1}{3}-\frac{1}{3}}{8-1} = 0$	
4	5		$\frac{4-2}{8-2} = \frac{1}{3}$		$\frac{\frac{-1}{16}-0}{10-1} = -\frac{1}{144}$
		$\frac{21-5}{8-4} = 4$		$\frac{\frac{-1}{6}-\frac{1}{3}}{10-2} = -\frac{1}{16}$	
8	21		$\frac{3-4}{10-4} = \frac{1}{6}$		
		$\frac{27-21}{10-8} = 3$			
10	27				

Newton's divided difference formula is given by

$$f(x) = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0 + (x - x_0)(x - x_1)(x - x_2)(x - x_3)\Delta^4 y_0 + \cdots$$

Differentiating w.r.t. x, we get

$$f^1(x) = \Delta y_0 + [(x - x_0) + (x - x_1)]\Delta^2 y_0 + [(x - x_0)(x - x_1) + (x - x_1)(x - x_2) + (x - x_0)(x - x_2)]\Delta^3 y_0 + [(x - x_0)(x - x_1)(x - x_2) + (x - x_1)(x - x_2)(x - x_3) + (x - x_2)(x - x_3)(x - x_0) + (x - x_3)(x - x_0)(x - x_1)]\Delta^4 y_0$$

$$f^1(4) = 1 + [(4-1) + (4-2)]\left(\frac{1}{3}\right) + [(4-1)(4-2) + (4-2)(4-4) + (4-1)(4-4)](0)$$
$$+ [(4-1)(4-2)(4-4) + (4-2)(4-4)(4-8) + (4-4)(4-8)(4-1)$$
$$+ (4-8)(4-1)(4-2)]\left(-\frac{1}{144}\right) = 1 + 1.66667 + 0 + 0.166667 = 2.833$$

$$f^1(5) = 1 + [(5-1)+(5-2)]\left(\frac{1}{3}\right) + [(5-1)(5-2)+(5-2)(5-4)+(5-1)(5-4)](0)$$
$$+ [(5-1)(5-2)(5-4)+(5-2)(5-4)(5-8)+(5-4)(5-8)(5-1)$$
$$+ (5-8)(5-1)(5-2)]\left(-\frac{1}{144}\right) = 1+4+\frac{27}{144} = 5+0.1875 = 5.1875$$

19. From the following table, find the value of x for which y is maximum and find this value of y.

x	1.2	1.3	1.4	1.5	1.6
y	0.9320	0.9636	0.9855	0.9975	0.9996

Sol) The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.2	0.9320				
		0.0316			
1.3	0.9636		-0.0097		
		0.0219		-0.0002	
1.4	0.9855		-0.0099		0.0002
		0.0120		0	
1.5	0.9975		-0.0099		
		0.0021			
1.6	0.9996				

$x_0 = 1.2,\ h = 0.1$ by Newton's forward differences formula, terminated after second differences, given

$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 = 0.932 + u(0.0316) + \frac{u(u-1)}{2!}(-0.0097) \to (1)$

$\Rightarrow \frac{dy}{du} = 0.0136 + \frac{2u-1}{2}(-0.0097)$ For y to be maximum, $\frac{dy}{du} = 0 \Rightarrow 2(0.0136) = (2u-1)(0.0097)$

$$\Rightarrow 2u-1 = \frac{0.0632}{0.0097} = 6.51546 \Rightarrow u = 3.7577\ \ hence\ x = x_0 + uh = 1.2 + (3.7577)(0.1) = 1.5758$$

$\therefore y\ is\ maximum\ when\ x = 1.5758 \approx 1.58$

Putting $u = 3.7577\ in\ (1)$, the maximum value of y

$= 0.9320 +$ (3.7577)(0.0316)+(3.7577)(3.7577-1)(-0.0097)/2 = 0.932+0.11874-0.0502586=1.00048=1.00

20. Find the maximum and minimum values of the function from following table.

x	0	1	2	3	4	5
y	0	0.25	0	2.25	16	56.25

Sol) The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	0				
		0.25			
1	0.25		-0.50		
		-0.25		3	
2	0		2.5		6
		2.25		9	
3	2.25		11.25		6
		13.75		15	
4	16		26.50		
		40.25			
5	56.25				

$x_0 = 0,\ h = 1$ by Newton's forward differences formula is $y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 +$

Differentiating above w.r.t. x $y^1 = \frac{1}{h}[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{24}\Delta^4 y_0 + \cdots$

$$= 0.25 + \frac{2u-1}{2}(-0.5) + \frac{3u^2-6u+2}{6}(3) + \frac{4u^3-18u^2+22u-6}{24}(6)$$

$$= \frac{1}{4} + \left(\frac{-u+0.5}{2}\right) + \frac{3u^2-6u+2}{2} + \frac{2u^3-9u^2+11u-3}{2}$$

$$= \frac{1}{4}[1 - 2u + 1 + 6u^2 - 12u + 4 + 4u^3 - 18u^2 + 22u - 6] = \frac{1}{4}[4u^3 - 12u^2 + 8u] = u^3 - 3u^2 + 2u$$

For y to be maximum/minimum, $\frac{dy}{du} = 0 \Rightarrow u^3 - 3u^2 + 2u = 0 \Rightarrow u(u-1)(u-2) = 0 \Rightarrow u = 0,1,2$

$For x_0 = 0, h = 1\ and\ \ x = x_0 + uh \Rightarrow x = 0,1,2\ \ but \frac{d^2y}{du^2} = 3u^2 - 6u + 2\ \ \ now \frac{d^2y}{du^2} > 0\ when\ u = 0,2$

$and \frac{d^2y}{du^2} < 0\ when\ u = 1$

$\therefore At\ x = 0,2;\ f(x)\ has\ minimum\ and\ minimum\ values\ are\ f(0) = 0, f(2) = 0\ and$

$At\ x = 1; f(x)\ has\ maximum\ and\ maximum\ values\ are\ f(1) = 0.25$

21. From the following table, find the value of x for which y is maximum and find this value of y.

x	3	4	5	6	7	8
y	0.205	0.240	0.259	0.262	0.250	0.224

Sol) The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
3	0.205					
		0.035				
4	0.240		-0.016			
		0.019		0		
5	0.259		-0.016		0.001	
		0.003		0.001		-0.001
6	0.262		-0.015		0	
		-0.012		0.001		
7	0.250		-0.014			
		-0.026				
8	0.224					

$x_0 = 3,\ h = 1$ by Newton's forward differences formula, terminated after second differences, given

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 = 0.205 + u(0.035) + \frac{u(u-1)}{2!}(-0.016) \to (1)$$

$$\Rightarrow \frac{dy}{du} = 0.035 + \frac{2u-1}{2}(-0.016)$$ For y to be maximum, $\frac{dy}{du} = 0 \Rightarrow 2(0.035) = (2u-1)(0.016)$

$$\Rightarrow 2u - 1 = \frac{0.07}{0.016} = 4.375 \Rightarrow u = 2.6875\ hence\ x = x_0 + uh = 3 + (2.6875)(1) = 5.6875$$

$$\therefore y\ is\ maximum\ when\ x = 5.6875 \approx 5.69$$

Putting $u = 2.6875\ in\ (1)$, the maximum value of y

= 0.205 + (2.6875)(0.035)+(2.6875)(2.6875-1)(-0.016)/2 = 0.2628

22. From the following table, find the value of x for which y is minimum and find this value of y.

x	0	2	4	6
y	3	3	11	27

Sol) The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	3			
		0		
2	3		8	
		8		0
4	11		8	
		16		
6	27			

$x_0 = 0,\ h = 2$ by Newton's forward differences formula, terminated after second differences, given

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 = 3 + u(0) + \frac{u(u-1)}{2!}(8) \to (1)$$

$\Rightarrow \frac{dy}{du} = 1(0) + \frac{2u-1}{2}(8)$ For y to be minimum, $\frac{dy}{du} = 0 \Rightarrow (2u-1) = 0 \Rightarrow 2u - 1 = 0 \Rightarrow u = 0.5\ hence\ x = x_0 + uh = 0 + (0.5)(2) = 1$

$\therefore y\ is\ minimum\ when\ x = 1$ Putting $u = 0.5\ in\ (1)$, the minimum value of y

= 3 + (0.5) (0) + (0.5) (0.5-1) (8)/2 = 3+0-1=2

23. From the following table, find the value of x for which y is minimum and find this value of y.

x	0.60	0.65	0.70	0.75
y	0.6221	0.6155	0.6138	0.6170

Sol) The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.60	0.6221			
		-0.0066		
0.65	0.6155		0.0049	
		-0.0017		0
0.70	0.6138		0.0049	
		0.0032		
0.75	0.6170			

$x_0 = 0.6,\ h = 0.05$ by Newton's forward differences formula, terminated after second differences, given

$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 = 0.6221 + u(-0.0066) + \frac{u(u-1)}{2!}(0.0049) \rightarrow (1)$

$\Rightarrow \frac{dy}{du} = 1(-0.0066) + \frac{2u-1}{2}(0.0049)$ For y to be minimum, $\frac{dy}{du} = 0 \Rightarrow 2(0.0066) = (2u-1)(0.0049)$

$\Rightarrow 2u - 1 = \frac{0.0132}{0.0049} \Rightarrow 2u - 1 = 2.6939 \Rightarrow u = 1.847\ hence\ x = x_0 + uh = 0.6 + (1.847)(0.05) = 0.6924$

$\therefore y\ is\ minimum\ when\ x = 0.6924$ Putting $u = 1.847\ in\ (1)$, the minimum value of y

= 0.6 + (0.05)(-0.0066)+(0.05)(0.05-1)(0.0049)/2 = 0.6-0.00033-0.000116375= 0.5996

Numerical integration

Numerical integration is the process of evaluating a definite integral $\int_a^b f(x)dx$ from a set of known numerical values of the integrand $f(x)$. If it is applied to the integration of a single variable function, the process is known as quadrature.

General quadrature formula for equidistant ordinates: - Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \dots y_n$ corresponding to the values of the argument $x_0 = a, x_1 = a + h, x_2 = a + 2h, \dots x_n = a + nh = b \ \ so\ that\ b - a = nh$. Then $\int_a^b ydx = h[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \cdots]$

Proof) Divide the intervals [a b] into n subintervals of width h such that $x_0 = a, x_1 = a + h, \dots x_n = a + nh = b$

Approximating y by Newton's forward difference formula, we get

$$\int_a^b ydx = \int_{x_0}^{x_n = x_0 + nh} [y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots]dx$$

Since $u = \frac{x - x_0}{h} \Longrightarrow uh = x - x_0 \Longrightarrow x = x_0 + uh \Longrightarrow dx = 0 + hdu \Longrightarrow dx = hdu$

$$\int_a^b ydx = \int_{u=0}^{n} [y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0 + \cdots]hdu$$

$$= h\int_{u=0}^{n}[y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0 + \cdots]\, du$$

$$= h\left[uy_0 + \frac{u^2}{2}\Delta y_0 + \left(\frac{u^3}{3} - \frac{u^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{u^4}{4} - \frac{3n^3}{3} + 2\frac{u^2}{2}\right)\frac{\Delta^3 y_0}{3!} + \cdots\right]_0^n$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \cdots\right]$$

$$\int_a^b ydx = h[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \cdots]$$

The Trapezoidal rule: - Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \dots y_n$ corresponding to the values of the argument $x_0 = a, x_1 = a + h, x_2 = a + 2h, \dots x_n = a + nh = b \ \ so\ that\ b - a = nh$. Then

$$\int_a^b ydx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})]$$

Proof) Divide the intervals [a b] into n subintervals of width h such that $x_0 = a, x_1 = a + h, \dots x_n = a + nh = b$

The general quadrature formula is

$$\int_{x_0}^{x_n} ydx = h[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \cdots] \to (1)$$

Put n=1 in (1) and neglecting second and higher order differences,

we get $\int_{x_0}^{x_1} y\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = h[y_0 + \frac{1}{2}(y_0 - y_1) = \frac{h}{2}[y_0 + y_1] \rightarrow (2)$

For the next intervals $[x_1\ x_2]$ we get $\int_{x_1}^{x_2} y\,dx = \frac{h}{2}[y_0 + y_1] \rightarrow (3)$

For the last intervals $[x_{n-1}\ x_n]$ we get $\int_{x_{n-1}}^{x_n} y\,dx = \frac{h}{2}[y_{n-1} + y_n] \rightarrow (4)$

Adding all these expression, we get

$$\int_{x_0}^{x_1} y\,dx + \int_{x_1}^{x_2} y\,dx + \cdots + \int_{x_{n-1}}^{n} y\,dx = \frac{h}{2}[y_0 + y_1] + \frac{h}{2}[y_1 + y_2] + \cdots + \frac{h}{2}[y_{n-1} + y_n]$$

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}[y_0 + y_1 + y_1 + y_2 + \cdots + y_{n-1} + y_n]$$

$$\int_a^b ydx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})]$$

Simpson's 1/3 rd rule: - Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \ldots y_n$ corresponding to the values of the argument $x_0 = a, x_1 = a + h, x_2 = a + 2h, \ldots x_n = a + nh = b\ \ so\ that\ b - a = nh\ and\ n\ is\ even.$ Then

$$\int_a^b ydx = \frac{h}{3}[(y_0 + y_n) + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1}]$$

Proof) Divide the intervals [a b] into n subintervals of width h such that $x_0 = a, x_1 = a + h, \ldots x_n = a + nh = b\ \ so\ that\ n\ is\ even.$

The general quadrature formula is

$$\int_{x_0}^{x_n} ydx = h[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \cdots] \rightarrow (1)$$

Put n=2 in (1) and neglecting third and higher order differences,

$$\int_{x_0}^{x_2} y\,dx = h\left[2y_0 + 2\Delta y_0 + \frac{1}{3}\Delta^2 y_0\right] = h\left[2y_0 + 2(y_1 - y_0) + \frac{1}{3}(y_2 - 2y_1 + y_0)\right] = \frac{h}{3}[y_0 + 4y_1 + y_2] \rightarrow (2)$$

Similarly $\int_{x_2}^{x_4} y\,dx = \frac{h}{3}[y_2 + 4y_3 + y_4]$ ... $\int_{x_{n-1}}^{x_n} y\,dx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$

Adding all these integrals, we get

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}[y_0 + 4y_1 + y_2 + y_2 + 4y_3 + y_4 + y_{n-2} + 4y_{n-1} + y_n]$$

$$\int_a^b ydx = \frac{h}{3}[(y_0 + y_n) + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1}]$$

Simpson's 3/8 th rule: - Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \dots y_n$ corresponding to the values of the argument $x_0 = a, x_1 = a + h, x_2 = a + 2h, \dots x_n = a + nh = b\ \ so\ that\ b - a = nh\ and\ n\ is\ a\ multiple\ of\ 3$. Then $\int_a^b y dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots + y_{n-3}]$

Proof) Divide the intervals [a b] into n subintervals of width h such that $x_0 = a, x_1 = a + h, \dots x_n = a + nh = b\ \ so\ that\ n\ is\ a\ multiple\ of\ 3$.

The general quadrature formula is

$$\int_{x_0}^{x_n} ydx = h[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \cdots] \to (1)$$

Put n=3 in (1) and neglecting fourth and higher order differences,

$$\int_{x_0}^{x_3} y\,dx = h\left[3y_0 + \frac{9}{2}\Delta y_0 + \left(\frac{27}{3} - \frac{9}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{81}{4} - 27 + 9\right)\frac{\Delta^3 y_0}{3!}\right] = h[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0) = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3] \to (2)$$

Similarly $\int_{x_3}^{x_6} y\,dx = \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6]$ ... $\int_{x_{n-3}}^{x_n} y\,dx = \frac{3h}{8}\ [y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$

Adding all these integrals, we get

$$\int_{x_0}^{x_n} ydx = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3 + y_3 + 3y_4 + 3y_5 + y_6 + \cdots + y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

$$\therefore \int_a^b ydx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots + y_{n-3}]$$

Weddle's rule: - Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \dots y_n$ corresponding to the values of the argument $x_0 = a, x_1 = a + h, x_2 = a + 2h, \dots x_n = a + nh = b\ \ so\ that\ b - a = nh\ and\ n\ is\ a\ multiple\ of\ 6$. Then $\int_a^b ydx = \frac{3h}{10}[(y_0 + y_2 + y_4 + y_8 + \cdots) + 5(y_1 + y_5 + y_7 + \cdots) + 6(y_3 + y_9 + \cdots) + 2y_6]$

Proof) Divide the intervals [a b] into n subintervals of width h such that $x_0 = a, x_1 = a + h, \dots x_n = a + nh = b\ \ so\ that\ n\ is\ a\ multiple\ of\ 6$.

The general quadrature formula is

$$\int_{x_0}^{x_n} ydx = h[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \cdots] \to (1)$$

Put n=6 in (1) and neglecting seventh and higher order differences,

$$\int_{x_0}^{x_6} y\,dx = h\left[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{41}{140}\Delta^6 y_0\right]$$

$$= \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

Similarly $\int_{x_6}^{x_{12}} y\, dx = \frac{3h}{10}[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}]$ ….

$$\int_{x_{n-6}}^{x_n} y\, dx = \frac{3h}{10}[y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n]$$

Adding all these integrals, we get

$$\int_{x_0}^{x_n} ydx = \frac{3h}{8}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \cdots + y_n]$$

$$\therefore \int_a^b ydx = \frac{3h}{10}[(y_0 + y_2 + y_4 + y_8 + \cdots) + 5(y_1 + y_5 + y_7 + \cdots) + 6(y_3 + y_9 + \cdots) + 2y_6]$$

Trapezoidal rule:

1. Derive Trapezoidal rule?

2. Evaluate $\int_0^5 \frac{dx}{4x+5}$ by using Trapezoidal rule.

Sol) Here $a = 0, b = 5, n = 5$ and $y = f(x) = \frac{1}{4x+5}$ $\therefore h = \frac{b-a}{n} = \frac{5-0}{5} = 1$

The values of x and y are tabulated below:

x	0	1	2	3	4	5
$y = \frac{1}{4x+5}$	0.2	0.1111	0.0769	0.0588	0.0476	0.04
	y_0	y_1	y_2	y_3	y_4	y_5

By Trapezoidal rule,

$$\int_0^5 \frac{dx}{4x+5} = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})] = \frac{h}{2}[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)]$$

$$= \frac{1}{2}[(0.2 + 0.04) + 2(0.1111 + 0.0769 + 0.0588 + 0.0476)] = 0.2672$$

2. Find the value of $\int_3^7 x^2 \, logx dx$ by taking4 strips and using trapezoidal rule.

Sol) Here $a = 3, b = 7, n = 4$ and $y = f(x) = x^2 logx$ $\therefore h = \frac{b-a}{n} = \frac{7-3}{4} = 1$

The values of x and y are tabulated below:

x	3	4	5	6	7
$y = x^2 logx$	9.8875	22.1807	40.2359	64.5033	95.3496
	y_0	y_1	y_2	y_3	y_4

By Trapezoidal rule,

$$\int_{3}^{7} x^2 logxdx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})] = \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{1}{2}[(9.8875 + 95.3496) + 2(22.1807 + 40.2359 + 64.5033)] = 179.53845$$

3. Apply trapezoidal rule by dividing the range into 6 equal parts to find value of $\int_0^{\pi} sinxdx$

Sol) Here $a = 0, b = \pi, n = 6$ and $y = f(x) = sinx$ $\quad \therefore h = \frac{b-a}{n} = \frac{\pi - 0}{6} = \frac{\pi}{6}$

The values of x and y are tabulated below:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$y = sinx$	0	0.5	0.866	1	0.866	0.5	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Trapezoidal rule,

$$\int_{0}^{\pi} sinxdx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})] = \frac{\pi}{12}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{\pi}{12}[(0 + 0) + 2(0.5 + 0.866 + 1 + 0.866 + 0.5)] = 1.954$$

4. Evaluate $\int_0^1 (4x - 3x^2)dx$ taking 10 intervals by trapezoidal rule.

Sol) Here $a = 0, b = 2, n = 10 and$ $y = f(x) = 4x - 3x^2$ $\quad \therefore h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1$

The values of x and y are tabulated below:

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
y	0	0.37	0.68	0.93	1.12	1.25	1.32	1.33	1.28	1.17	1
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By Trapezoidal rule,

$$\int_0^1 (4x - 3x^2)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})]$$
$$= \frac{0.1}{2}[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)]$$
$$= \frac{0.1}{2}[(0 + 1) + 2(0.37 + 0.68 + 0.93 + 1.12 + 1.25 + 1.32 + 1.33 + 1.28 + 1.17)]$$
$$= 0.05[1 + 2(9.45)] = 0.995$$

5. Evaluate $\int_{-2}^{2} \frac{t}{5+2t} dt$ by using trapezoidal rule with nine points (i.e n=8 strips).

Sol) Here $a = -2, b = 2, n = 8 and\ y = f(t) = \frac{t}{5+2t}$ $\therefore h = \frac{b-a}{n} = \frac{2-(-2)}{8} = 0.5$

The values of x and y are tabulated below:

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
y	-2	-0.75	-0.333	-0.125	0	0.08333	0.14286	0.1875	0.2222
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By Trapezoidal rule,

$$\int_{-2}^{2} \left(\frac{t}{5 + 2t}\right) dt = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})]$$
$$= \frac{0.1}{2}[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$
$$= \frac{0.5}{2}[(-2 + 0.2222) + 2(-0.75 - 0.333 - 0.125 + 0 + 0.08333 + 0.14286 + 0.1875)]$$
$$= -0.8416$$

6. Evaluate $\int_0^{\pi/2} sinxdx$ also compare with its exact value.

Sol) Here $a = 0, b = \frac{\pi}{2}, n = 3\ and\ y = f(x) = sinx$ $\therefore h = \frac{b-a}{n} = \frac{\frac{\pi}{2}-0}{3} = \frac{\pi}{6}$

The values of x and y are tabulated below:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$y = sinx$	0	0.5	0.866	1
	y_0	y_1	y_2	y_3

By Trapezoidal rule,

$$\int_0^{\pi} sinxdx = \frac{h}{2}[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})] = \frac{\pi}{12}[(y_0+y_3)+2(y_1+y_2)]$$

$$= \frac{\pi}{12}[(0+1)+2(0.5+0.866)] = 0.9774$$

Exact value of $\int_0^{\pi/2} sinxdx = -[cosx]_0^{\pi/2} = -(0-1) = 1$

Hence Error = Exact value – Approximate value = 1.000-0.9774 = 0.0226

7. Evaluate $\int_0^6 \frac{1}{1+x^2}dx$ by trapezoidal rule.

Sol) Here $a=0, b=6, n=6 and\ y=f(x)=\frac{1}{1+x^2}$ $\quad \therefore h=\frac{b-a}{n}=\frac{6-0}{6}=1$

The values of x and y are tabulated below:

x	0	1	2	3	4	5	6
y	1	0.5	0.2	0.1	0.0588	0.0385	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Trapezoidal rule,

$$\int_0^6 \left(\frac{1}{1+x^2}\right)dx = \frac{h}{2}[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})] = \frac{1}{2}[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)]$$

$$= \frac{1}{2}[(1+0.027)+2(0.5+0.2+0.1+0.0588+0.0385] = \frac{1}{2}(2.8216) = 1.4108$$

8. Evaluate $\int_0^1 \frac{1}{1+x^2}dx$ by trapezoidal rule.

Sol) Here $a=0, b=1, n=6\ and\ y=f(x)=\frac{1}{1+x}$ $\quad \therefore h=\frac{b-a}{n}=\frac{1-0}{6}=\frac{1}{6}$

The values of x and y are tabulated below:

x	0	1/6	2/6	3/6	4/6	5/6	6/6
y	1	0.85714	0.75	0.66667	0.6	0.54545	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) By trapezoidal rule,

$$\int_0^1 \left(\frac{1}{1+x^2}\right)dx = \frac{h}{2}[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})] = \frac{1}{12}[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)]$$

$$= \frac{1}{12}[(1+0.5)+2(0.85714+0.75+0.66667+0.6+0.54545)] = \frac{1}{12}(8.33852) = 0.6949$$

9. Evaluate $\int_0^1 \frac{1}{1+x} dx$ by trapezoidal rule with h=0.125

Sol) Here $a = 0, b = 1, n = 8 and\ y = f(x) = \frac{1}{1+x}$ $\therefore h = \frac{b-a}{n} = \frac{1-0}{8} = 0.125$

The values of $x\ and\ y$ are tabulated below:

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	1	0.8889	0.8	0.7273	0.6667	0.6154	0.5714	0.5333	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By Trapezoidal rule, $\int_0^1 \left(\frac{1}{1+x}\right) dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})]$

$$= \frac{0.125}{2}[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.125}{2}[(1 + 0.5) + 2(0.8889 + 0.8 + 0.7273 + 0.6667 + 0.6154 + 0.5714 + 0.5333)] = 0.69413$$

10. Evaluate $\int_0^\pi tsint dt$ using the trapezoidal rule.

Sol) Here $a = 0, b = \pi, n = 6\ and\ y = f(t) = tsint$ $\therefore h = \frac{b-a}{n} = \frac{\pi-0}{6} = \frac{\pi}{6}$

The values of $x\ and\ y$ are tabulated below:

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π
$y = tsint$	0	0.2618	0.9069	1.5708	1.8138	1.309	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Trapezoidal rule,

$$\int_0^\pi tsintdt = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})] = \frac{\pi}{12}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{\pi}{12}[(0 + 0) + 2(0.2618 + 0.9069 + 1.5708 + 1.8138 + 1.309)] = 3.0695$$

11. From the following table, find the area bounded by the curve and the x-axis from x =7.47 to x =7.52

x	7.47	7.48	7.49	7.50	7.51	7.52
$y = f(x)$	1.93	1.95	1.98	2.01	2.03	2.06
	y_0	y_1	y_2	y_3	y_4	y_5

Sol) Here h =0.01 and n =5. So we cannot use Simpson's rule(both) or Weddle's rule. Hence we will use Trapezoidal rule.

By Trapezoidal rule, we have

Required area $=\int_{7.47}^{7.52} ydx = \frac{h}{2}[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})] = \frac{0.01}{2}[(y_0+y_5)+2(y_1+y_2+y_3+y_4)] = \frac{0.01}{2}[(1.93+2.06)+2(1.95+1.98+2.01+2.03)] = \frac{0.01}{2}(19.93) = 0.09965$

12. Evaluate $\int_0^4 e^x dx$ using Trapezoidal rule.

Sol) Here $a=0, b=4, n=4\ and\ y=f(x)=e^x$ $\quad \therefore h=\frac{b-a}{n}=\frac{4-0}{4}=1$

The values of $x\ and\ y$ are tabulated below:

x	0	1	2	3	4
$y=e^x$	1	2.71828	7.3890	20.0855	54.5981
	y_0	y_1	y_2	y_3	y_4

By Trapezoidal rule, $\int_0^4 e^x dx = \frac{h}{2}[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})] = \frac{1}{2}[(y_0+y_4)+2(y_1+y_2+y_3)] = \frac{1}{2}[(1+54.5981)+2(2.71828+7.389+20.0855)] = 57.992$

13. Evaluate $\int_0^1 cosxdx$ using h =0.2 by trapezoidal rule

Sol) Here $a=0, b=1, n=5\ and\ y=f(x)=cosx$ $\quad \therefore h=\frac{b-a}{n}=\frac{1-0}{5}=0.2$

The values of $x\ and\ y$ are tabulated below:

x	0	0.2	0.4	0.6	0.8	1
$y=cosx$	1	0.9801	0.9211	0.8253	0.6967	0.5403
	y_0	y_1	y_2	y_3	y_4	y_5

By Trapezoidal rule,

$$\int_0^1 cosxdx = \frac{h}{2}[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})] = \frac{0.2}{2}[(y_0+y_5)+2(y_1+y_2+y_3+y_4)]$$

$$= \frac{0.2}{2}[(1+0.5403)+2(0.9801+0.9211+0.8253+0.6967)] = 0.8387$$

14. Evaluate $\int_0^{\pi/2} \sqrt{cosx}dx$ by divided the interval $[0, \frac{\pi}{2}]$ into 6 points.

Sol) Here $a=0, b=\frac{\pi}{2}, n=6\ and\ y=f(x)=\sqrt{cosx}$ $\quad \therefore h=\frac{b-a}{n}=\frac{\pi/2-0}{6}=\frac{\pi}{12}$

The values of $x\ and\ y$ are tabulated below:

x	0	$\frac{\pi}{12}$	$\frac{2\pi}{12}$	$\frac{3\pi}{12}$	$\frac{4\pi}{12}$	$\frac{5\pi}{12}$	$\frac{6\pi}{12}$
$y = \sqrt{cosx}$	1	0.9828	0.9306	0.8408	0.7071	0.5087	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Trapezoidal rule,

$$\int_0^{\pi/2} \sqrt{cosx}dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})] = \frac{\pi}{24}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{\pi}{24}[(1 + 0) + 2(0.9828 + 0.9306 + 0.8408 + 0.7071 + 0.5087)] = 1.1697$$

15. Evaluate $\int_1^2 \frac{1}{x^2} dx$ by using trapezoidal rule with h =0.1

Sol) Here $a = 1, b = 2, n = 10 and\ y = f(x) = \frac{1}{x^2}$ $\quad \therefore h = \frac{b-a}{n} = \frac{2-1}{10} = 0.1$

The values of $x\ and\ y$ are tabulated below:

x	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
y	1	0.8264	0.6944	0.5917	0.5102	0.4444	0.3906	0.3460	0.3086	0.2770	0.25
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By Trapezoidal rule, $\int_0^1 \left(\frac{1}{x^2}\right) dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})]$

$$= \frac{0.1}{2}[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)]$$

$$= \frac{0.1}{2}[(1 + 0.25) + 2(4.3893)] = 0.5014$$

16. Evaluate $\int_1^5 logxdx$ taking 8 sub intervals correct to 4 decimal places by trapezoidal rule

Sol) Here $a = 1, b = 5, n = 8 and\ y = f(x) = log10^x$ $\quad \therefore h = \frac{b-a}{n} = \frac{5-1}{8} = 0.5$

The values of $x\ and\ y$ are tabulated below:

x	1	1.5	2	2.5	3	3.5	4	4.5	5
y	0	0.176	0.3010	0.3979	0.4771	0.5440	0.6020	0.6532	0.6989
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By Trapezoidal rule, $\int_0^1 (log)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})]$

$$= \frac{0.5}{2}[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.5}{2}[(1 + 0.6989) + 2(3.1502)] = 1.9998$$

Simpson's $\frac{1}{3}$ *rd rule*:

1. Evaluate $\int_1^3 \frac{1}{x} dx$ by Simpson's $\frac{1}{3}$ $rd\ rule\ with\ n = 4\ and\ n = 8$ subintervals respectively. Determine the error by direct integration.

Sol) Here $a = 1, b = 3, n = 4\ and\ y = f(x) = \frac{1}{x}$ $\quad \therefore h = \frac{b-a}{n} = \frac{3-1}{4} = 0.5$

The values of $x\ and\ y$ are tabulated below:

x	1	15	2	2.5	3
$y = \frac{1}{x}$	1	0.66667	0.5	0.4	0.33333
	y_0	y_1	y_2	y_3	y_4

Simpson's $\frac{1}{3}$ $rd\ rule\ is$ $\int_1^3 \frac{1}{x} dx = \frac{h}{3}[(y_0 + y_n) + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1}]$

$$= \frac{h}{3}[(y_0 + y_4) + 2(y_2) + 4(y_1 + y_3)] = \frac{0.5}{3}[(1 + 0.33333) + 2(0.5) + 4(0.66667 + 0.4)] = 1.100002$$

$$a = 1, b = 3, n = 8\ and\ y = f(x) = \frac{1}{x} \qquad \therefore h = \frac{b-a}{n} = \frac{3-1}{8} = 0.25$$

The values of $x\ and\ y$ are tabulated below:

x	1	1.25	1.5	1.75	2	2.25	2.5	2.75	3
y	1	0.8	0.66667	0.57143	0.5	0.44444	0.4	0.36364	0.33333
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Simpson's $\frac{1}{3}$ $rd\ rule\ is$ $\int_1^3 \frac{1}{x} dx = \frac{h}{3}[(y_0 + y_n) + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1}]$

$= \frac{h}{3}[(y_0 + y_8) + 2(y_2 + y_4 + y_6 + y_8) + 4(y_1 + y_3 + y_5 + y_7) = \frac{0.25}{3}[(1 + 0.33333) + 2(0.66667 + 0.5 + 0.4 + 0.33333) + 4(0.8 + 0.57143 + 0.44444 + 0.36364)] = 1.09873$

By direct integration, we get $\int_1^3 \frac{1}{x} dx = log3 - log1 = 1.098612$

Error for h =0.5 is $|1.098612 - 1.100002| = 0.00139$ and for h =0.25 is$|1.098612 - 1.09873| = 0.000118$

2. Evaluate $\int_0^1 \frac{1}{1+x^2} dx$ by using Simpson's $\frac{1}{3}$ $rd\ rule$ with 10 equal strips and deduce an approximation value of π.

Sol) Here $a=0, b=1, n=10 and\ y=f(x)=\frac{1}{1+x^2}$ $\therefore h=\frac{b-a}{n}=\frac{1-0}{10}=0.1$

The values of $x\ and\ y$ are tabulated below:

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
y	1	0.9901	0.96153	0.91743	0.86207	0.8	0.73529	0.67114	0.60976	0.55249	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

Simpson's $\frac{1}{3}\ rd\ rule\ is \int_0^1 \frac{1}{1+x^2}dx=\frac{h}{3}[(y_0+y_n)+2(y_2+y_4+\cdots+y_{n-2})+4(y_1+y_3+\cdots+y_{n-1}]$

$=\frac{h}{3}[(y_0+y_{10})+2(y_2+y_4+y_6+y_8)+4(y_1+y_3+y_5+y_7+y_9)=\frac{0.1}{3}[(1+0.5)+2(0.0.96153+0.86207+0.73529+0.60976)+4(0.9901+091743+0.8+0.67114+0.55249)]=0.785398$

By direct integration, we get $\int_0^1 \frac{1}{1+x^2}dx=[\tan^{-1}x]_0^1=\tan^{-1}1-\tan^{-1}0=\frac{\pi}{4}$

$\therefore \frac{\pi}{4}=0.785398 \Rightarrow \pi \approx 3.141592$

3. Evaluate $\int_0^1 e^x dx$ approximately in steps of 0.05 using Simpson's 1/3 rd rule

Sol) Here $a=0, b=1, n=20 and\ y=f(x)=e^x$ $\therefore h=\frac{b-a}{n}=\frac{1-0}{20}=0.05$

The values of $x\ and\ y$ are tabulated below:

x	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
y	1	1.0513	1.1052	1.1618	1.2214	1.2840	1.3499	1.4191	1.4918	1.5683	1.6487
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
x	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1	
y	1.7333	1.8221	1.9155	2.0138	2.1170	2.2255	2.3396	2.4596	2.5857	2.7183	
	y_{11}	y_{12}	y_{13}	y_{14}	y_{15}	y_{16}	y_{17}	y_{18}	y_{19}	y_{20}	

Simpson's $\frac{1}{3}\ rd\ rule\ is \int_0^1 e^x dx=\frac{h}{3}[(y_0+y_n)+2(y_2+y_4+\cdots+y_{n-2})+4(y_1+y_3+\cdots+y_{n-1}]$

$=\frac{h}{3}[(y_0+y_{20})+2(y_2+y_4+y_6+\cdots+y_{18})+4(y_1+y_3+y_5+y_7+\cdots+y_{19})$

$$=\frac{0.05}{3}[(1+2.7183)+2(15.3380)+4(17.1796)]=1.7185$$

4. Evaluate $\int_0^1 \frac{1}{1+x}dx$ by using Simpson's 1/3 rd rule

Sol) We divide the interval [0 1] into 6 (multiple of 3) subintervals So $a=0, b=1, n=6\ and$

$$y=f(x)=\frac{1}{1+x} \qquad \therefore h=\frac{b-a}{n}=\frac{1-0}{20}=0.05$$

The values of $x \; and \; y$ are tabulated below:

x	0	1/6	2/6	3/6	4/6	5/6	1
y	1	0.8571	0.75	0.6666	0.6	0.5454	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Simpson's $\frac{1}{3}$ $rd \; rule \; is \int_0^1 \frac{1}{1+x} dx = \frac{h}{3}[(y_0 + y_n) + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1}]$

$= \frac{h}{3}[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5) = \frac{1}{18}[(1 + 0.5) + 2(0.75 + 0.6) + 4(0.8571 + 0.6666 + 0.5454)] = 0.6931$

By direct integration, we get $\int_0^1 \frac{1}{1+x} dx = [\log(1+x)]_0^1 = log2 - log1 = 0.6931$

5. Evaluate $\int_0^{\pi/2} \sqrt{sin\theta} d\theta$ by using Simpson's 1/3 rd rule with h $= \frac{\pi}{8}$

Sol) $a = 0, b = \frac{\pi}{2}, \; y = f(\theta) = \sqrt{sin\theta} \qquad \therefore h = \frac{\pi}{8}$

The values of $x \; and \; y$ are tabulated below:

x	0	$\frac{\pi}{8}$	$\frac{2\pi}{8}$	$\frac{3\pi}{8}$	$\frac{4\pi}{8}$
y	0	0.6186	0.8409	0.9612	1
	y_0	y_1	y_2	y_3	y_4

Simpson's $\frac{1}{3}$ $rd \; rule \; is \int_0^{\pi/2} \sqrt{sin\theta} d\theta = \frac{h}{3}[(y_0 + y_n) + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1}]$

$= \frac{h}{3}[(y_0 + y_4) + 2(y_2) + 4(y_1 + y_3) = \frac{\pi}{24}[(0 + 1) + 2(0.8409) + 4(0.6186 + 0.9612)] = 1.1782$

6. Evaluate $\int_0^4 e^x dx$ using Simpson's 1/3 rd rule

Sol) Here $a = 0, b = 4, n = 4 \; and \; y = f(x) = e^x \qquad \therefore h = \frac{b-a}{n} = \frac{4-0}{4} = 1$

The values of $x \; and \; y$ are tabulated below:

x	0	1	2	3	4
$y = e^x$	1	2.71828	7.3890	20.0855	54.5981
	y_0	y_1	y_2	y_3	y_4

Simpson's $\frac{1}{3}$ $rd \; rule \; is \int_0^4 e^x = \frac{h}{3}[(y_0 + y_n) + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1}]$

$= \frac{h}{3}[(y_0 + y_4) + 2(y_2) + 4(y_1 + y_3) = \frac{1}{3}[(1 + 54.5981) + 2(7.3890) + 4(2.71828 + 20.0855)]$

$= 53.8637$

7. Evaluate $\int_1^7 \frac{1}{x} dx$ using Simpson's 1/3 rd rule rule

Sol) Here $a = 1, b = 7, n = 6$ and $y = f(x) = \frac{1}{x}$ $\therefore h = \frac{b-a}{n} = \frac{7-1}{6} = 1$

The values of x and y are tabulated below:

x	1	2	3	4	5	6	7
$y = \frac{1}{x}$	1	0.5	0.3333	0.25	0.2	0.1667	0.1429
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Simpson's $\frac{1}{3}$ rd $rule$ is $\int_1^7 \frac{1}{x} dx = \frac{h}{3}[(y_0 + y_n) + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1}]$

$= \frac{h}{3}[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5) = \frac{1}{3}[(1 + 0.1429) + 2(0.3333 + 0.2) + 4(0.5 + 0.25 + 0.1667)] = \frac{1}{3}(5.8763) = 1.9587$

$The\ Exact\ value\ of\ \int_1^7 \frac{1}{x} dx = [logx]_1^7 = log7 = 1.9587$

8. Evaluate $\int_1^2 \sqrt{x - \frac{1}{x}} dx$ by using Simpson's 1/3 rd rule with 5 ordinates

Sol) $a = 1, b = 2,\ n = 4,\ y = f(x) = \sqrt{x - \frac{1}{x}}$ $\therefore h = \frac{b-a}{n} = \frac{1}{4} = 0.25$

The values of x and y are tabulated below:

x	1	1.25	1.5	1.75	2
y	0	0.6708	0.9129	1.0856	1.2247
	y_0	y_1	y_2	y_3	y_4

Simpson's $\frac{1}{3}$ rd $rule$ is $\int_1^2 \sqrt{x - \frac{1}{x}} dx = \frac{h}{3}[(y_0 + y_n) + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1}]$

$= \frac{h}{3}[(y_0 + y_4) + 2(y_2) + 4(y_1 + y_3) = \frac{0.25}{3}[(0 + 1.2247) + 2(0.9129) + 4(0.6708 + 1.0856)] = 0.8397$

Simpson's $\frac{3}{8}$ *th rule*

1. Find approximate value of $\int_0^{\pi/2} \sqrt{cos\theta} d\theta$ by Simpson's$\frac{3}{8}$ $th\ rule$, dividing the interval into six parts

Sol) Here $a = 0, b = \frac{\pi}{2}, n = 6$ and $y = f(\theta) = \sqrt{cos\theta}$ $\therefore h = \frac{b-a}{n} = \frac{\frac{\pi}{2}-0}{6} = \frac{\pi}{12}$

The values of x and y are tabulated below:

x	0	$\frac{\pi}{12}$	$\frac{2\pi}{12}$	$\frac{3\pi}{12}$	$\frac{4\pi}{12}$	$\frac{5\pi}{12}$	$\frac{6\pi}{12}$
$y=\sqrt{cos\theta}$	1	0.98282	0.93060	0.84090	0.70711	0.50874	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Simpson's $\frac{3}{8}$ $th\ rule\ is \int_0^{\pi/2}\sqrt{cos\theta}d\theta=\frac{3h}{8}[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots+y_{n-1})+2(y_3+y_6+y_9+\cdots+y_{n-3}]$

$=\frac{3h}{8}[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)=\frac{\pi}{32}[(1+0)+3(0.98282+0.93060+0.70711+0.50874)+2(0.84090)]=1.185408$

2. Use the Simpson's $\frac{3}{8}$ $th\ rule$ to obtain an approximate value of $\int_0^{0.3}(1-8x^3)^{\frac{1}{2}}dx.$

Sol) Here $a=0,b=0.3,n=3\ and\ y=f(x)=\sqrt{(1-8x^3)}$ $\therefore h=\frac{b-a}{n}=\frac{0.3-0}{3}=0.1$

The values of $x\ and\ y$ are tabulated below:

x	0	0.1	0.2	0.3
y	1	0.9960	0.96747	0.88544
	y_0	y_1	y_2	y_3

Simpson's $\frac{3}{8}$ $th\ rule\ is \int_0^{0.3}\sqrt{(1-8x^3}dx=\frac{3h}{8}[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots+y_{n-1})+2(y_3+y_6+y_9+\cdots+y_{n-3}]$

$=\frac{3h}{8}[(y_0+y_3)+3(y_1+y_2)]=\frac{0.3}{8}[(1+0.88544)+3(0.9960+0.96747)]=0.291594$

3. Evaluate $\int_0^6\frac{1}{1+x^2}dx$ by using Simpson's $\frac{3}{8}$ $th\ rule$

Sol) Here $a=0,b=6,n=6\ and\ y=f(x)=\frac{1}{1+x^2}$ $\therefore h=\frac{b-a}{n}=\frac{6-0}{6}=1$

The values of $x\ and\ y$ are tabulated below:

x	0	1	2	3	4	5	6
y	1	0.5	0.2	0.1	0.0588	0.0385	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Simpson's $\frac{3}{8}$ $th\ rule\ is \int_0^6\frac{1}{1+x^2}dx=\frac{3h}{8}[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots+y_{n-1})+2(y_3+y_6+y_9+\cdots+y_{n-3}]$

$=\frac{3}{8}[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)=\frac{3}{8}[(1+0.27)+3(0.5+0.2+0.0588+0.0385)+2(0.1)]=1.3732$

4. Obtain an approximate value of π from the equation $\frac{\pi}{4} = \int_0^1 \frac{1}{1+x^2} dx$ using Simpson's $\frac{3}{8}$ *th rule with* 9 *ordinates.*

Sol) Here $a = 0, b = 1, n = 8$ *and* $y = f(x) = \frac{1}{1+x^2}$ $\quad \therefore h = \frac{b-a}{n} = \frac{1-0}{8} = 0.125$

The values of x *and* y are tabulated below:

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	1	0.9846	0.9412	0.8767	0.8	0.7191	0.64	0.5664	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Simpson's $\frac{3}{8}$ *th rule is* $\int_0^1 \frac{1}{1+x^2} dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots + y_{n-3}]$

$= \frac{3}{8}[(y_0 + y_8) + 3(y_1 + y_2 + y_4 + y_5 + y_7) + 2(y_3 + y_6) = \frac{3(0.125)}{8}[(1 + 0.5) + 3(0.9846 + 0.9412 + 0.8 + 0.7191 + 0.5664) + 2(0.8767 + 0.64)] = \frac{0.375}{8}[16.5673] = 0.77655$

By actual integration, $\int_0^1 \frac{1}{1+x^2} dx = [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}$

$$\therefore \frac{\pi}{4} = 0.77659219 \Rightarrow \pi = 3.1062$$

5. Obtain an approximate value of π from the equation $\frac{\pi}{4} = \int_0^1 \frac{1}{1+x^2} dx$ using Simpson's$\frac{1}{3}$ *and* $\frac{3}{8}$ *th rule with* 7 *ordinates.*

Sol) Here $a = 0, b = 1, n = 6$ *and* $y = f(x) = \frac{1}{1+x^2}$ $\quad \therefore h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$

The values of x *and* y are tabulated below:

x	0	1/6	2/6	3/6	4/6	5/6	6/6
y	1	0.9729	0.9	0.8	0.6923	0.5901	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Simpson's $\frac{1}{3}$ *rd rule is* $\int_0^1 \frac{1}{1+x^2} dx = \frac{h}{3}[(y_0 + y_n) + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1}]$

$= \frac{h}{3}[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5) = \frac{1}{18}[(1 + 0.5) + 2(0.9 + 0.6923) + 4(0.9729 + 0.8 + 0.5901)] = \frac{1}{18}[14.1366] = 0.785366$

By direct integration, we get $\int_0^1 \frac{1}{1+x^2} dx = [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}$

$$\therefore \frac{\pi}{4} = 0.785366 \Rightarrow \pi \approx 3.141464$$

Simpson's $\frac{3}{8}$ $th\ rule\ is \int_0^1 \frac{1}{1+x^2} dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots + y_{n-3}]$

$= \frac{3}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3) = \frac{1}{16}[(1 + 0.5) + 3(0.9729 + 0.9 + 0.6923 + 0.5901) + 2(0.8)] = \frac{1}{16}[12.5659] = 0.785368$

By actual integration, $\int_0^1 \frac{1}{1+x^2} dx = [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}$

$$\therefore \frac{\pi}{4} = 0.785368 \Rightarrow \pi \approx 3.141472$$

6. Using Simpson's $\frac{3}{8}$ $th\ rule, evaluate\ \int_0^1 \frac{1}{1+x} dx\ with\ h = \frac{1}{6}$

Sol) Here $a = 0, b = 1, n = 6\ and\ y = f(x) = \frac{1}{1+x}$ $\therefore h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$

The values of $x\ and\ y$ are tabulated below:

x	0	1/6	2/6	3/6	4/6	5/6	6/6
y	1	0.85714	0.75	0.66667	0.6	0.54545	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Simpson's $\frac{3}{8}$ $th\ rule\ is \int_0^1 \frac{1}{1+x} dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots + y_{n-3}]$

$= \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3) = \frac{1}{16}[(1 + 0.5) + 3(0.85714 + 0.75 + 0.6 + 0.54545) + 2(0.66667)] = \frac{1}{16}[11.09111] = 0.69319$

7. Using Simpson's $\frac{3}{8}$ $th\ rule, evaluate\ \int_0^1 \frac{1}{1+x} dx$ by dividing the interval [0 1] into 10 parts

Sol) Here $a = 0, b = 1, n = 10\ and\ y = f(x) = \frac{1}{1+x}$ $\therefore h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1$

The values of $x\ and\ y$ are tabulated below:

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
y	1	0.9090	0.8333	0.7692	0.7143	0.6667	0.625	0.5882	0.5556	0.5263	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

Simpson's $\frac{3}{8}$ $th\ rule\ is \int_0^1 \frac{1}{1+x} dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots + y_{n-3}]$

$= \frac{3h}{8}[(y_0 + y_{10}) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + y_8) + 2(y_3 + y_6 + y_9) = \frac{0.3}{8}[(1 + 0.5) + 3(0.9090 + 0.8333 + 0.7143 + 0.6667 + 0.5882 + 0.5556) + 2(0.7692 + 0.625 + 0.5263)] = \frac{0.3}{8}[18.1423] = 0.6803$

8. Evaluate by Simpson's $\frac{3}{8}$ $th\ rule,\ \int_1^2 \frac{2x}{1+x^4}dx\ with\ h = 0.25$

Sol) Here $a = 1, b = 2, n = 4\ and\ y = f(x) = \frac{2x}{1+x^4}$ $\quad \therefore h = \frac{b-a}{n} = \frac{2-1}{4} = 0.25$

The values of $x\ and\ y$ are tabulated below:

x	1	1.25	1.5	1.75	2
y	1	0.7264	0.4948	0.3372	0.2353
	y_0	y_1	y_2	y_3	y_4

Simpson's $\frac{3}{8}$ $th\ rule\ is\ \int_1^2 \frac{2x}{1+x^4}\ dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots + y_{n-3}]$

$= \frac{3h}{8}[(y_0 + y_4) + 3(y_1 + y_2) + 2(y_3)] = \frac{0.75}{8}[(1 + 0.2353) + 3(0.7264 + 0.4948) + 2(0.3372)] = 0.478$

9. Find the value of $\int_3^7 x^2\ logxdx$ by taking4 strips and using Simpson's $\frac{3}{8}$ $th\ rule$.

Sol) Here $a = 3, b = 7, n = 4\ and\ y = f(x) = x^2 logx$ $\quad \therefore h = \frac{b-a}{n} = \frac{7-3}{4} = 1$

The values of $x\ and\ y$ are tabulated below:

x	3	4	5	6	7
$y = x^2 logx$	9.8875	22.1807	40.2359	64.5033	95.3496
	y_0	y_1	y_2	y_3	y_4

Simpson's $\frac{3}{8}$ $th\ rule\ is\ \int_3^7 x^2 logx\ \ dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots + y_{n-3}]$

$= \frac{3h}{8}[(y_0 + y_4) + 3(y_1 + y_2) + 2(y_3)] = \frac{3}{8}[(9.8875 + 95.3496) + 3(22.1807 + 40.2359) + 2(64.5033)]$

$= 158.06$

Weddle's rule:

1. Derive Weddle's rule?

2. Apply Weddle's rule to evaluate $\int_0^6 \frac{1}{1+x^2}dx$ by dividing the range into 6 parts.

Sol) Here $a = 0, b = 6, n = 6\ and\ y = f(x) = \frac{1}{1+x^2}$ $\quad \therefore h = \frac{b-a}{n} = \frac{6-0}{6} = 1$

The values of $x\ and\ y$ are tabulated below:

x	0	1	2	3	4	5	6
y	1	0.5	0.2	0.1	0.0588	0.0385	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Weddle's rule,

$$\int_0^6 \frac{1}{1+x^2}dx = \frac{3h}{10}[(y_0 + y_2 + y_4 + y_8 + \cdots) + 5(y_1 + y_5 + y_7 + \cdots) + 6(y_3 + y_9 + \cdots) + 2y_6]$$

$$= \frac{3h}{10}[(y_0 + y_6) + 5(y_1 + y_5) + (y_2 + +y_4) + 6y_3]$$

$$= \frac{3}{10}[(1 + 0.027) + 5(0.5 + 0.0385) + (0.2 + 0.0588) + 6(0.1) = 0.3[4.5783] = 1.3734$$

3. Apply Weddle's rule to evaluate $\int_0^1 \frac{1}{1+x^2}dx$ by taking h =1/6

Sol) Here $a = 0, b = 1, n = 6$ and $y = f(x) = \frac{1}{1+x^2}$ $\therefore h = \frac{b-a}{6} = \frac{1-0}{6} = \frac{1}{6}$

The values of x and y are tabulated below:

x	0	1/6	2/6	3/6	4/6	5/6	6/6
y	1	0.9729	0.9	0.8	0.6923	0.5901	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Weddle's rule,

$$\int_0^6 \frac{1}{1+x^2}dx = \frac{3h}{10}[(y_0 + y_2 + y_4 + y_8 + \cdots) + 5(y_1 + y_5 + y_7 + \cdots) + 6(y_3 + y_9 + \cdots) + 2y_6]$$

$$= \frac{3h}{10}[(y_0 + y_6) + 5(y_1 + y_5) + (y_2 + +y_4) + 6y_3]$$

$$= \frac{1}{20}[(1 + 0.5) + 5(0.9729 + 0.5901) + (0.9 + 0.6923) + 6(0.8) = \frac{1}{20}[15.773] = 0.78865$$

4. Evaluate $\int_4^{5.2} logxdx, using\ weddle's\ ru\ le$.

Sol) Here $a = 4, b = 5.2, n = 6$ and $y = f(x) = logx$ $\therefore h = \frac{b-a}{6} = \frac{5.2-4}{6} = 0.2$

The values of x and y are tabulated below:

x	4	4.2	4.4	4.6	4.8	5	5.2
y	1.3862	1.4350	1.4816	1.5261	1.5686	1.6094	1.6486
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Weddle's rule,

$$\int_4^{5.2} logx dx = \frac{3h}{10}[(y_0 + y_2 + y_4 + y_8 + \cdots) + 5(y_1 + y_5 + y_7 + \cdots) + 6(y_3 + y_9 + \cdots) + 2y_6]$$

$$= \frac{3(0.2)}{10}[(y_0 + y_6) + 5(y_1 + y_5) + (y_2 + +y_4) + 6y_3]$$

$$= \frac{0.6}{10}[(1.3862 + 1.6486) + 5(1.4350 + 1.6094) + (1.4816 + 1.5686) + 6(1.5261) = \frac{0.6}{10}[30.4636]$$

$$= 1.8278$$

5. Evaluate $\int_0^1 \frac{1}{1+x} dx$ by using Weddle's rule

Sol) Here $a = 0, b = 1, n = 6$ and $y = f(x) = \frac{1}{1+x}$ $\quad \therefore h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$

The values of x and y are tabulated below:

x	0	1/6	2/6	3/6	4/6	5/6	6/6
y	1	0.85714	0.75	0.66667	0.6	0.54545	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Weddle's rule,

$$\int_0^1 \frac{1}{1+x} dx = \frac{3h}{10}[(y_0 + y_2 + y_4 + y_8 + \cdots) + 5(y_1 + y_5 + y_7 + \cdots) + 6(y_3 + y_9 + \cdots) + 2y_6]$$

$$= \frac{1}{20}[(y_0 + y_6) + 5(y_1 + y_5) + (y_2 + +y_4) + 6y_3]$$

$$= \frac{1}{20}[(1 + 0.5) + 5(0.85714 + 0.54545) + (0.75 + 0.6) + 6(0.66667) = \frac{1}{20}[13.86297] = 0.6932$$

6. Evaluate $\int_0^{12} y dx$ using Weddle's rule from the following table

x	0	2	4	6	8	10	12
y	0	22	30	27	18	7	0

Sol) we have h = 2

x	0	2	4	6	8	10	12
y	0	22	30	27	18	7	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Weddle's rule,

$$\int_0^{12} y dx = \frac{3h}{10}[(y_0 + y_2 + y_4 + y_8 + \cdots) + 5(y_1 + y_5 + y_7 + \cdots) + 6(y_3 + y_9 + \cdots) + 2y_6]$$

$$= \frac{3h}{10}[(y_0 + y_6) + 5(y_1 + y_5) + (y_2 + +y_4) + 6y_3]$$

$$= \frac{6}{10}[(0 + 0) + 5(22 + 7) + (30 + 18) + 6(27) = \frac{3}{5}[355] = 213$$

Errors in numerical integration:

1) The error in Trapezoidal rule is $= -\frac{(x_n - x_0)}{12} h^2 f^{11}(\xi) \quad where\ x_0 < \xi < x_n$

The error in the trapezoidal rule is of the order h^2

2) The error in Simpson's 1/3 rd rule is $= -\frac{(x_n - x_0)}{180} h^4 f^4(\xi) \quad where\ x_0 < \xi < x_n$

The error in the Simpson's 1/3 rd rule is of the order h^4

3) The error in Simpson's 3/8 th rule is $= -\frac{(x_n - x_0)}{80} h^4 f^4(\xi) \quad where\ x_0 < \xi < x_n$

The error in the Simpson's 3/8 th rule is of the order h^4

4) The error in Weddle's rule is $= -\frac{(x_n - x_0)}{840} h^6 f^6(\xi) \quad where\ x_0 < \xi < x_n$

The error in the Weddle's rule is of the order h^6

1. Calculate $\int_0^1 \frac{1}{1+x} dx$ using trapezoidal rule, Simpson's 1/3, 3/8 rules and Weddle's rule and compute the errors.

Sol) For trapezoidal rule, the range is to be divided into any number of equal parts, for Simpson's 1/3. 3/8 rule and Weddle's rule the range is to be divided into even, multiple of 3 and multiple of 6 equal parts.

Here $a = 0, b = 1, n = 6\ and\ y = f(x) = \frac{1}{1+x}$ $\quad \therefore h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$

The values of $x\ and\ y$ are tabulated below:

x	0	1/6	2/6	3/6	4/6	5/6	6/6
y	1	0.85714	0.75	0.66667	0.6	0.54545	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) By trapezoidal rule,

$$\int_0^1 \left(\frac{1}{1+x^2}\right) dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})] = \frac{1}{12}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{1}{12}[(1+0.5)+2(0.85714+0.75+0.66667+0.6+0.54545)] = \frac{1}{12}(8.33852) = 0.6949$$

The error in Trapezoidal rule is $= -\frac{(x_n - x_0)}{12} h^2 f^{11}(\xi) \quad where\ x_0 < \xi < x_n$

$$= \frac{-1}{12}\left(\frac{1}{6}\right)^2 \frac{2}{(1+\xi)^3} \qquad \left[\because f^{11}(x) = \frac{2}{(1+x)^3}\right] where\ x_0 < \xi < x_n = 0 < \xi < 1$$

When $\xi = 0\ then\ the\ error\ is\ \frac{1}{12}\left(\frac{1}{6}\right)^2 \frac{2}{(1+\xi)^3} = \frac{1}{12}\left(\frac{1}{6}\right)^2 \frac{2}{(1+0)^3} = 0.004629$

When $\xi = 1\ then\ the\ error\ is\ \frac{1}{12}\left(\frac{1}{6}\right)^2 \frac{2}{(1+\xi)^3} = \frac{1}{12}\left(\frac{1}{6}\right)^2 \frac{2}{(1+1)^3} = 0.0005787$

Hence the error lies in between 0.0005787 and 0.004629

(ii) Simpson's $\frac{1}{3}\ rd\ rule\ is \int_0^1 \frac{1}{1+x}dx = \frac{h}{3}[(y_0 + y_n) + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1}]$

$= \frac{h}{3}[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5) = \frac{1}{18}[(1+0.5) + 2(0.75+0.6) + 4(0.8571 + 0.6666 + 0.5454)] = 0.6931$

The error in Simpson's 1/3 rd rule is $= -\frac{(x_n - x_0)}{180} h^4 f^4(\xi) \quad where\ x_0 < \xi < x_n$

$$= \frac{-1}{180}\left(\frac{1}{6}\right)^4 \frac{24}{(1+\xi)^5} \qquad \left[\because f^4(x) = \frac{24}{(1+x)^5}\right] where\ x_0 < \xi < x_n = 0 < \xi < 1$$

When $\xi = 0\ then\ the\ error\ is\ \frac{1}{180}\left(\frac{1}{6}\right)^4 \frac{24}{(1+\xi)^5} = \frac{1}{180}\left(\frac{1}{6}\right)^4 \frac{24}{(1+0)^5} = \frac{1}{180 \times 54} = 0.0010$

When $\xi = 1\ then\ the\ error\ is\ \frac{1}{180}\left(\frac{1}{6}\right)^4 \frac{24}{(1+\xi)^5} = \frac{1}{180}\left(\frac{1}{6}\right)^4 \frac{24}{(1+1)^3} = \frac{1}{180 \times 5432} = 0.00003$

Hence the error lies in between 0.00003 and 0.0010

(iii) Simpson's $\frac{3}{8}\ th\ rule\ is \int_0^1 \frac{1}{1+x}dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots + y_{n-3}]$

$= \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3) = \frac{1}{16}[(1+0.5) + 3(0.85714 + 0.75 + 0.6 + 0.54545) + 2(0.66667)] = \frac{1}{16}[11.09111] = 0.6932$

The error in Simpson's 3/8 th rule is $= -\frac{(x_n - x_0)}{80} h^4 f^4(\xi) \quad where\ x_0 < \xi < x_n$

$$= \frac{-1}{80}\left(\frac{1}{6}\right)^4 \frac{24}{(1+\xi)^5} \qquad \left[\because f^4(x) = \frac{24}{(1+x)^5}\right] where\ x_0 < \xi < x_n = 0 < \xi < 1$$

When $\xi = 0\ then\ the\ error\ is\ \frac{1}{80}\left(\frac{1}{6}\right)^4 \frac{24}{(1+\xi)^5} = \frac{1}{80}\left(\frac{1}{6}\right)^4 \frac{24}{(1+0)^5} = \frac{1}{80 \times 54} = 0.0002$

When $\xi = 1$ *then the error is* $\frac{1}{80}\left(\frac{1}{6}\right)^4 \frac{24}{(1+\xi)^5} = \frac{1}{80}\left(\frac{1}{6}\right)^4 \frac{24}{(1+1)^3} = \frac{1}{80\times 5432} = 0.00006$

Hence the error lies in between 0.00006 and 0.0002

(iv) By Weddle's rule,

$$\int_0^1 \frac{1}{1+x}dx = \frac{3h}{10}[(y_0 + y_2 + y_4 + y_8 + \cdots) + 5(y_1 + y_5 + y_7 + \cdots) + 6(y_3 + y_9 + \cdots) + 2y_6]$$

$$= \frac{1}{20}[(y_0 + y_6) + 5(y_1 + y_5) + (y_2 + +y_4) + 6y_3]$$

$$= \frac{1}{20}[(1 + 0.5) + 5(0.85714 + 0.54545) + (0.75 + 0.6) + 6(0.66667) = \frac{1}{20}[13.86297] = 0.6932$$

The error in Weddle's rule is $= -\frac{(x_n - x_0)}{840} h^6 f^6(\xi) \quad where\ x_0 < \xi < x_n$

$$= \frac{-1}{840}\left(\frac{1}{6}\right)^6 \frac{720}{(1+\xi)^7} \qquad \left[\because f^7(x) = \frac{720}{(1+x)^7}\right] where\ x_0 < \xi < x_n = 0 < \xi < 1$$

When $\xi = 0$ *then the error is* $\frac{1}{840}\left(\frac{1}{6}\right)^6 \frac{720}{(1+\xi)^7} = \frac{1}{840}\left(\frac{1}{6}\right)^6 \frac{720}{(1+0)^7} = \frac{1}{840\times 64.8} = 0.00001$

When $\xi = 1$ *then the error is* $\frac{1}{840}\left(\frac{1}{6}\right)^6 \frac{720}{(1+\xi)^7} = \frac{1}{840}\left(\frac{1}{6}\right)^6 \frac{720}{(1+1)^7} = \frac{1}{840\times 64.8\times 128} = 0.0000001$

Hence the error lies in between 0.0000001 and 0.000001

Numerical solution of ordinary differential equations

Numerical solution of ordinary differential equations: -The solution of an ordinary differential equation in which x is the independent variable and y is the dependent variable usually means an explicite for expression for y in terms of a finite number of elementary functions of x i.e. polynomial, trigonometric or an exponential function. If such an explicit relation is found then the solution is known as the finite form of solution. In the absence of such solution we need a numerical method of solution.

Methods: -

1. Taylor's series
2. Picard's method of successive approximations
3. Euler's method
4. Modified Euler's method
5. Runge-Kutta method

. **Taylor's series method:** -To find the numerical solution of differential equation $\frac{dy}{dx} = f(x, y) \rightarrow (1)\ with\ y(x_0) = y_0$

$y(x)$ can be expanded about the point x_0 in a Taylor's series in powers of $(x - x_0)$ as

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!} y^1(x_0) + \frac{(x - x_0)^2}{2!} y^{11}(x_0) + \frac{(x - x_0)^3}{3!} y^{111}(x_0) + \frac{(x - x_0)^4}{4!} y^{1V}(x_0) + \cdots \rightarrow (2)$$

Where $y^i(x_0)$ is the i^{th} derivative of $y(x)\ at\ x = x_0$. Substitute these values in (1)

If we let $(x - x_0) = h$ $(i.e. x = x_1 = x_0 + h$ we can write the Taylor's series as

$$y(x) = y(x_0) + \frac{h}{1!} y^1(x_0) + \frac{h^2}{2!} y^{11}(x_0) + \frac{h^3}{3!} y^{111}(x_0) + \frac{h^4}{4!} y^{1V}(x_0) + \cdots$$

$$i.e. y_1 = y(x_0) + \frac{h}{1!} y^1(x_0) + \frac{h^2}{2!} y^{11}(x_0) + \frac{h^3}{3!} y^{111}(x_0) + \frac{h^4}{4!} y^{1V}(x_0) + \cdots$$

Picard's method of successive approximation method: -

Consider the differential equation $\frac{dy}{dx} = f(x, y) \rightarrow (1)\ with\ y(x_0) = y_0 \rightarrow (2)$

It is required to obtain the sol of (1) subject to the condition (2)

The equation is $dy = f(x, y)dx$. Integrating this equation in the interval (x_0, x) we get

$$\int_{x=x_0}^{x} dy = \int_{x_0}^{x} f(x, y)dx \Rightarrow (y)_{x=x_0}^{x} = \int_{x_0}^{x} f(x, y)dx \Rightarrow y(x) - y(x_0) = \int_{x_0}^{x} f(x, y)dx$$

$\Rightarrow y(x) = y(x_0) + \int_{x_0}^{x} f(x,y)dx \to (3)$ put $y = y_0$ in R.H.S of (3) we get

$\Rightarrow y_1(x) = y(x_0) + \int_{x_0}^{x} f(x,y_0)dx$ put $y = y_1$ in R.H.S of (3) we get

$$\Rightarrow y_2(x) = y(x_0) + \int_{x_0}^{x} f(x,y_1)dx$$

$$\vdots$$

$$\Rightarrow y_{n+1}(x) = y(x_0) + \int_{x_0}^{x} f(x,y_n)dx \quad where\ n = 0,1,2,3,\dots$$

Euler's method: - Consider the differential equation $\frac{dy}{dx} = f(x,y)\ with\ y = y_0\ when\ x = x_0 \to (1)$

To solve (1) for the values of y at $x = x_i$ where $x_i = x_0 + ih \quad i = 1,2,3,\dots$

Integrating (1) we get $y_1 = y_0 + \int_{x_0}^{x} f(x,y)dx$

In the range $x_0 \le x \le x_1$ put $f(x,y) = f(x_0,y_0)$ in above equation

$y_1 = y_0 + \int_{x_0}^{x_1} f(x_0,y_0)dx = y_0 + (x_1 - x_0)f(x_0,y_0) = y_0 + hf(x_0,y_0) \quad (use\ x_1 - x_0 = h)$

Similarly in the range $x_1 \le x \le x_2$ we have $y_2 = y_1 + hf(x_1,y_1)$

Proceeding in this way finally we have $y_{n+1} = y_n + hf(x_n,y_n) \quad where\ n = 0,1,2,\dots$

The Euler formula is $y_{n+1} = y_n + hf(x_n,y_n) \quad where\ n = 0,1,2,\dots$

Note: -Euler's method is Runge-Kutta method of first order.

Modified Euler's method: - Consider the differential equation $\frac{dy}{dx} = f(x,y)\ with\ y = y_0$

$when\ x = x_0 \to (1)$

Integrating (1) from $x_0\ to\ x_1$ we get $\int_{x_0}^{x_1} dy = \int_{x_0}^{x_1} f(x.y) \Rightarrow y(x_1) - y(x_0) = \int_{x_0}^{x_1} f(x,y)\,dx$

$y(x_1) = y(x_0) + \int_{x_0}^{x_1} f(x,y)\,dx$

Approximating the integral by trapezoidal rule we get $y_1 = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1,y_1)]$ where $x_1 - x_0 = h$

$$y_1^{(0)} = y_0 + hf(x_0,y_0)$$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1,y_1^{(0)})]$$

$$y_1{}^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1{}^{(1)})]$$

$\vdots$

$y_1{}^{(n+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1{}^{(n)})]$ $where\ n = 0,1,2, \ldots.$ If two successive values of $y_1{}^{(n)}, y_1{}^{(n+1)}$ are close to one another, we will take the common value as y_1. now we have $\frac{dy}{dx} = f(x, y)$ $with\ y = y_1\ when\ x = x_1$. To get $y_1 = y(x_2) = y(x_1 + h)$ we use the above procedure again.

Modified Euler's method is

$$y_1{}^{(n+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1{}^{(n)})] \quad where\ n = 0,1,2, \ldots.$$

Note: - Modified Euler's method is Runge-Kutta method of second order

Runge-Kutta methods: -

First order Runge-Kutta method is $y_1 = y_0 + hk_1$ where $k_1 = f(x_0, y_0)$

Second order Runge-Kutta method is $y_1 = y_0 + \frac{1}{2}[k_1 + k_2]$

$where\ \ k_1 = hf(x_0, y_0) \quad and \quad k_2 = hf(x_0 + h, y_0 + k_1)$

Third order Runge-Kutta method is $y_1 = y_0 + \frac{1}{6}[k_1 + 4k_2 + k_3]$

$$where\ \ k_1 = hf(x_0, y_0) \quad and \quad k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \quad k_3 = h\,f(x_0 + h, y_0 + 2k_2 - k_1)$$

Fourth order Runge-Kutta method is $y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$$where\ \ k_1 = hf(x_0, y_0), \quad and \quad k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right),$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right), k_4 = hf(x_0 + h, y_0 + k_3)$$

Problems: -**Picard's method:**

1. Explain the Picard's method of successive approximation?

2. Solve $\frac{dy}{dx} = 2x - y\ with\ x_0 = 0, y_0 = 3$ by using Picard's method

Sol) Given $x_0 = 0,\ y_0 = 3$ and $\frac{dy}{dx} = 2x - y = f(x, y)$

Picard's formula is

$$y_{n+1}(x) = y(x_0) + \int_{x_0}^{x} f(x, y_n)dx \qquad where\ \ n = 0,1,2,3,\ldots \to (1)$$

Put $n = 0\ in\ (1) \Rightarrow y_1(x) = y_0 + \int_0^x f(x, y_0)dx \Rightarrow y_1(x) = 3 + \int_0^x (2x - y_0)dx$

$$\Rightarrow y_1(x) = 3 + \int_0^x (2x - 3)dx = 3 + 2\left[\frac{x^2}{2}\right]_0^x - 3[x]_0^x = x^2 - 3x + 3$$

Put $n = 1\ in\ (1) \Rightarrow y_2(x) = y_0 + \int_0^x f(x_0, y_1)dx \Rightarrow y_2(x) = 3 + \int_0^x (2x - y_1)dx$

$$\Rightarrow y_2(x) = 3 + \int_0^x (2x) - (x^2 - 3x + 3)dx$$

$$= 3 + \int_0^x (5x - x^2 - 3)dx = 3 + 5\left[\frac{x^2}{2}\right]_0^x - \left[\frac{x^3}{3}\right]_0^x - 3(x)_0^x \Rightarrow y_2(x) = \frac{5x^2}{2} - \frac{x^3}{3} - 3x + 3$$

Put $n = 2\ in\ (1) \Rightarrow y_3(x) = y_0 + \int_0^x f(x, y_2)dx \Rightarrow y_3(x) = 3 + \int_0^x (2x - y_2)dx$

$$\Rightarrow y_3(x) = 3 + \int_0^x (2x) - \left(\frac{5}{2}x^2 - \frac{x^3}{3} - 3x + 3\right)dx = 3 + \int_0^x (5x - \frac{5}{2}x^2 + \frac{1}{3}x^3 - 3)dx$$

$$= 3 + \left[5 \cdot \frac{x^2}{2} - \frac{5}{2} \cdot \frac{x^3}{3} + \frac{1}{3} \cdot \frac{x^4}{4} - 3\right]_0^x$$

$$\Rightarrow y_3(x) = \frac{5}{2}x^2 - \frac{5}{6}x^3 + \frac{1}{12}x^4 - 3x + 3$$

3. Solve $\frac{dy}{dx} = xy + 1\ with\ y(0) = 1$ by using Picard's method

Sol) Given $x_0 = 0,\ \ y_0 = 1$ and $\frac{dy}{dx} = xy + 1 = f(x, y)$

Picard's formula is

$$y_{n+1}(x) = y(x_0) + \int_{x_0}^{x} f(x, y_n)dx \qquad where\ \ n = 0,1,2,3,\ldots \to (1)$$

Put $n = 0\ in\ (1) \Rightarrow y_1(x) = y_0 + \int_0^x f(x, y_0)dx \Rightarrow y_1(x) = 1 + \int_0^x (xy_0 + 1)dx$

$$\Rightarrow y_1(x) = 1 + \int_0^x (x + 1)dx = 1 + \left[\frac{x^2}{2} + x\right]_0^x = 1 + x + \frac{x^2}{2}$$

Put $n = 1\ in\ (1) \Rightarrow y_2(x) = y_0 + \int_0^x f(x, y_1)dx \Rightarrow y_2(x) = 1 + \int_0^x (xy_1 + 1)dx$

$$\Rightarrow y_2(x) = 1 + \int_0^x [x\left(1 + x + \frac{x^2}{2}\right) + 1]dx = 1 + \int_0^x \left[\left(x + x^2 + \frac{x^3}{3}\right) + 1\right] dx = 1 + \left[\frac{x^2}{2} + \frac{x^3}{3} + \frac{1}{2}\frac{x^4}{4}\right]_0^x + x$$

$$\Rightarrow y_2(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{1}{8}x^4$$

Put $n = 2\ in\ (1) \Rightarrow y_3(x) = y_0 + \int_0^x f(x, y_2)dx \Rightarrow y_3(x) = 1 + \int_0^x (xy_2 + 1)dx$

$\Rightarrow y_3(x) = 1 + \int_0^x \left[x\left(1 + x + x^2 + \frac{x^3}{3} + \frac{1}{8}x^4\right) + 1\right] dx = 1 + \int_0^x (1 + x + x^2 + x^3 + \frac{1}{3}x^4 + \frac{1}{8}x^5)dx$

$\Rightarrow y_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{8}x^4 + \frac{1}{15}x^5 + \frac{1}{48}x^6$

4. Solve $\frac{dy}{dx} = x - y\ \ with\ y = 1\ when\ x = 0\ also\ find\ y(0.2)$ by using Picard's method

Sol) Given $x_0 = 0,\ y_0 = 1$ and $f(x, y) = x - y$

Picard's formula is

$$y_{n+1}(x) = y(x_0) + \int_{x_0}^x f(x, y_n)dx \qquad where\ \ n = 0,1,2,3, \dots \to (1)$$

Put $n = 0\ in\ (1) \Rightarrow y_1(x) = y_0 + \int_0^x f(x, y_0)dx \Rightarrow y_1(x) = 1 + \int_0^x (x - y_0)dx$

$$\Rightarrow y_1(x) = 1 + \int_0^x (x - 1)dx = 1 + \left[\frac{x^2}{2} - x\right]_0^x = 1 - x + \frac{x^2}{2}$$

Put $n = 1\ in\ (1) \Rightarrow y_2(x) = y_0 + \int_0^x f(x, y_1)dx \Rightarrow y_2(x) = 1 + \int_0^x (x - y_1)dx$

$$\Rightarrow y_2(x) = 1 + \int_0^x \left[x - \left(1 - x + \frac{x^2}{2}\right)\right] dx = 1 + \int_0^x \left[\left(2x - \frac{1}{2}x^2 - 1\right)\right] dx = 1 + \left[2\frac{x^2}{2} - \frac{x^3}{2(3)} - x\right]_0^x$$

$$\Rightarrow y_2(x) = 1 - x + x^2 - \frac{x^3}{6}$$

Put $n = 2\ in\ (1) \Rightarrow y_3(x) = y_0 + \int_0^x f(x, y_2)dx \Rightarrow y_3(x) = 1 + \int_0^x (x - y_2)dx$

$\Rightarrow y_3(x) = 1 + \int_0^x \left[x - \left(1 - x + x^2 - \frac{x^3}{6}\right)\right] dx = 1 + \int_0^x (-1 + 2x - x^2 + \frac{1}{6}x^3)dx$

$\Rightarrow y_3(x) = 1 - x + x^2 - \frac{1}{3}x^3 + \frac{1}{24}x^4$

When $x = 0.2$, the successive approximations of y are given by $y_0 = 1, y_1 = 0.82, y_2 = 0.83867$,

$y_3 = 0.834740$

5. Solve $\frac{dy}{dx} = y \;\; with\; y(0) = 1$ by Picard's method and compare the solution with the exact solution.

Sol) Given $x_0 = 0,\; y_0 = 1$ and $f(x, y) = y$

Picard's formula is

$$y_{n+1}(x) = y(x_0) + \int_{x_0}^{x} f(x, y_n)dx \qquad where\;\; n = 0,1,2,3,\ldots \to (1)$$

Put $n = 0\; in\; (1) \Rightarrow y_1(x) = y_0 + \int_0^x f(x, y_0)dx \Rightarrow y_1(x) = 1 + \int_0^x (y_0)dx$

$$\Rightarrow y_1(x) = 1 + \int_0^x (1)dx = 1 + x$$

Put $n = 1\; in\; (1) \Rightarrow y_2(x) = y_0 + \int_0^x f(x, y_1)dx \Rightarrow y_2(x) = 1 + \int_0^x (y_1)dx$

$$\Rightarrow y_2(x) = 1 + \int_0^x (1 + x)dx = 1 + x + \frac{1}{2}x^2$$

$$\Rightarrow y_2(x) = 1 + x + \frac{x^2}{2}$$

Put $n = 2\; in\; (1) \Rightarrow y_3(x) = y_0 + \int_0^x f(x, y_2)dx \Rightarrow y_3(x) = 1 + \int_0^x (y_2)dx$

$$\Rightarrow y_3(x) = 1 + \int_0^x \left(1 + x + \frac{x^2}{2}\right)dx = 1 + x + \frac{x^2}{2} + \frac{1}{2.3}x^3$$

$$similarly\;\; y_4(x) = 1 + x + \frac{x^2}{2!} + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$$

Exact solution: The given differential equation is

$$\frac{dy}{dx} = y \Rightarrow \frac{dy}{y} = dx \Rightarrow \int \frac{dy}{y} = \int dx \Rightarrow logy = x + c \Rightarrow y = e^{x+c}$$

$$\Rightarrow y = ke^x \;\; where\; k\; is\; an\; arbitrary\; constant$$

Substituting the initial values $x = 0, y = 1, we\; get\; k = 1$

The Exact solution is $y = e^x$ which has the expansion

$$y = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

The Picard's solution is same as the first five term of the exact solution.

6. Find the value of y for $x = 0.1$ by Picard's method given that $\frac{dy}{dx} = \frac{y-x}{y+x},\;\; y(0) = 1$

Sol) Given $x_0 = 0,\ y_0 = 1$ and $f(x,y) = \frac{y-x}{y+x}$

Picard's formula is

$$y_{n+1}(x) = y(x_0) + \int_{x_0}^{x} f(x, y_n)dx \qquad where\ \ n = 0,1,2,3, \ldots \to (1)$$

Put $n = 0\ in\ (1) \Rightarrow y_1(x) = y_0 + \int_0^x f(x, y_0)dx \Rightarrow y_1(x) = 1 + \int_0^x \left(\frac{1-x}{1+x}\right) dx = 1 + \int_0^x \left(-1 + \frac{2}{1+x}\right) dx$

$\Rightarrow y_1(x) == 1 + [-x + 2\log(1+x)]_0^x = 1 + [-x + 2\log(1+x)] = 1 - x + 2\log(1+x) \to (1)$

Put $n = 1\ in\ (1) \Rightarrow y_2(x) = y_0 + \int_0^x f(x, y_1)dx \Rightarrow y_2(x) = 1 + \int_0^x \left(\frac{1-x+2\log(1+x)-x}{1-x+2\log(1+x)+x}\right) dx$

$$\Rightarrow y_2(x) = 1 + \int_0^x \left[1 - \frac{2x}{1 + 2\log(1+x)}\right] dx = which\ is\ very\ difficult\ to\ integrate.$$

Hence we use the 1 st approximation. Taking $x = 0.1\ in$ (1), we get $y_1(0.1) = 1 - 0.1 + 2\log(1 + 0.1)$

$y_1(0.1) \approx y(0.1) = 1 - 0.1 + 2\log(1.1) = 0.9 + 2(0.0414) = 0.9828$

7. Solve the equation $y^1 = x + y^2$ subject to the condition $y = 1\ when\ x = 0$ using Picard's method.

Sol) Given $x_0 = 0,\ y_0 = 1$ and $f(x,y) = x + y^2$

Picard's formula is

$$y_{n+1}(x) = y(x_0) + \int_{x_0}^{x} f(x, y_n)dx \qquad where\ \ n = 0,1,2,3, \ldots \to (1)$$

Put $n = 0\ in\ (1) \Rightarrow y_1(x) = y_0 + \int_0^x f(x, y_0)dx \Rightarrow y_1(x) = 1 + \int_0^x (x + {y_0}^2)dx$

$$\Rightarrow y_1(x) = 1 + \int_0^x (x+1)dx = 1 + x + \frac{x^2}{2}$$

Put $n = 1\ in\ (1) \Rightarrow y_2(x) = y_0 + \int_0^x f(x, y_1)dx \Rightarrow y_2(x) = 1 + \int_0^x (x + {y_1}^2)dx$

$$\Rightarrow y_2(x) = 1 + \int_0^x \left[x + \left(1 + x + \frac{x^2}{2}\right)^2\right] dx = 1 + \int_0^x [\,x + \left(1 + x^2 + \frac{x^4}{4} + 2x + x^3 + x^2\right) dx$$

$$\Rightarrow y_2(x) = 1 + \int_0^x \left(1 + 3x + 2x^2 + x^3 + \frac{x^4}{4}\right) dx = 1 + x + \frac{3}{2}x^2 + \frac{2}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{20}x^5$$

8. Obtain Picard's second approximation solution of the initial value problem $\frac{dy}{dx} = \frac{x^2}{y^2+1}, \;\; y(0) = 0$

Sol) Given $x_0 = 0, \;\; y_0 = 0$ and $f(x,y) = \frac{x^2}{y^2+1}$

Picard's formula is

$$y_{n+1}(x) = y(x_0) + \int_{x_0}^{x} f(x, y_n)dx \qquad where \;\; n = 0,1,2,3,\dots \to (1)$$

Put $n = 0 \; in \; (1) \Rightarrow y_1(x) = y_0 + \int_0^x f(x, y_0)dx \Rightarrow y_1(x) = 0 + \int_0^x \left(\frac{x^2}{{y_0}^2+1}\right)dx = 0 + \int_0^x \left(\frac{x^2}{0+1}\right)dx$

$$\Rightarrow y_1(x) == 0 + \int_0^x x^2 dx = \frac{1}{3}x^3$$

Put $n = 1 \; in \; (1) \Rightarrow y_2(x) = y_0 + \int_0^x f(x, y_1)dx \Rightarrow y_2(x) = \int_0^x \left(\frac{x^2}{\left(\frac{x^3}{3}\right)^2+1}\right)dx$

$$to \; integrate \; , put \; \frac{x^3}{3} = t \Rightarrow 3x^2dx = 3dt \Rightarrow x^2dx = dt \quad when \; x = 0 \; then \; t = 0 \; and \; x = x \; then \; t = \frac{x^3}{3}$$

$$\Rightarrow y_2(x) = \int_0^{x^3/3} \left[\frac{x^2}{t^2+1}\right]\frac{dt}{x^2} = [\tan^{-1} t]_0^{x^3/3} = \tan^{-1}\left(\frac{x^3}{3}\right)$$

9. Obtain Picard's second approximation solution of the initial value problem $\frac{dy}{dx} = x^2 + y^2, \;\; y(0) = 0$

Sol) Given $x_0 = 0, \;\; y_0 = 0$ and $f(x,y) = x^2 + y^2$

Picard's formula is

$$y_{n+1}(x) = y(x_0) + \int_{x_0}^{x} f(x, y_n)dx \qquad where \;\; n = 0,1,2,3,\dots \to (1)$$

Put $n = 0 \; in \; (1) \Rightarrow y_1(x) = y_0 + \int_0^x f(x, y_0)dx \Rightarrow y_1(x) = 0 + \int_0^x (x^2 + {y_0}^2)dx$

$$\Rightarrow y_1(x) = \int_0^x (x^2 + 0)dx = \frac{x^3}{3}$$

Put $n = 1 \; in \; (1) \Rightarrow y_2(x) = y_0 + \int_0^x f(x, y_1)dx \Rightarrow y_2(x) = \int_0^x \left[x^2 + \left(\frac{x^3}{3}\right)^2\right]dx = \frac{x^3}{3} + \frac{x^7}{63}$

Put $n = 2 \; in \; (1) \Rightarrow y_3(x) = y_0 + \int_0^x f(x, y_2)dx$

$$\Rightarrow y_3(x) = \int_0^x \left[x^2 + \left(\frac{x^3}{3} + \frac{x^7}{63}\right)^2\right] dx = \frac{x^3}{3} + \frac{x^7}{63} + \frac{2x^{11}}{2079} + \frac{x^{15}}{59535}$$

Taylor's method:

1. Explain the method of solving the differential equation $\frac{dy}{dx} = f(x,y)\ with\ y(x_0) = y_0\ by\ Taylor's\ series\ method$

2. Solve $\frac{dy}{dx} = 1 + xy\ and\ \ y(0) = 1\ i.e.\ \ y_0 = 1$ using Taylor's series method and compute $y(0.1)$

Sol) Given $y^1 = 1 + xy \quad \to (1)$

Differentiating (1) w.r.to 'x' successively, we get

$y^{11} = xy^1 + y \to (2),\ \ y^{111} = xy^{11} + y^1 + y^1 = xy^{11} + 2y^1 \to (3)$

$y^{1v} = xy^{111} + y^{11} + 2y^{11} = xy^{111} + 3y^{11} \to (4)$ and so on.

We have $x_0 = 0, y_0 = 1$ substituting these values in (1),(2),(3),(4) we get

$y_0{}^1 = x_0y_0 + 1 = 0 + 1 = 1 \qquad y_0{}^{11} = x_0y_0{}^1 + y_0 = 0 + 1 = 1$

$y_0{}^{111} = x_0y_0{}^{11} + 2y_0{}^1 = 0 + 2(1) = 2 \qquad y_0{}^{1v} = x_0y_0{}^{111} + 3y_0{}^{11} = 0 + 3(1) = 3$

By Taylor's series formula is

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y^1(x_0) + \frac{(x-x_0)^2}{2!}y^{11}(x_0) + \frac{(x-x_0)^3}{3!}y^{111}(x_0) + \frac{(x-x_0)^4}{4!}y^{1V}(x_0) + \cdots$$

$$y(x) = y_0 + \frac{x}{1!}y_0{}^1 + \frac{x^2}{2!}y_0{}^{11} + \frac{x^3}{3!}y_0{}^{111} + \frac{x^4}{4!}y_0{}^{1V} + \cdots$$

$$y_1 = y(0.1) = 1 + (0.1) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(2) + \frac{(0.1)^4}{24}(3) + \cdots = 1 + 0.1 + 0.005 + 0.00033 + 0.0000125$$

$$= 1.1053425 \approx 1.1053$$

3. Evaluate $y(0.2)$ correct to four decimal places by Taylor's series method if $y^1 = 1 - 2xy\ \ with\ y(0) = 0$

Sol) Given $y^1 = 1 - 2xy \quad \to (1)$

Differentiating (1) w.r.to 'x' successively, we get

$y^{11} = -2xy^1 - 2y \to (2),\ \ y^{111} = -2xy^{11} - 2y^1 - 2y^1 = -2xy^{11} - 4y^1 \to (3)$

$y^{1v} = -2xy^{111} - 2y^{11} - 4y^{11} = -2xy^{111} - 6y^{11} \to (4)$ and so on.

We have $x_0 = 0, y_0 = 0$ substituting these values in (1),(2),(3),(4) we get

$y_0{}^1 = 1 - 2x_0y_0 = 1 - 0 = 1 \qquad y_0{}^{11} = -2x_0y_0{}^1 - 2y_0 = 0 + 0 = 0$

$y_0{}^{111} = -2x_0y_0{}^{11} - 4y_0{}^1 = 0 - 4(1) = -4 \qquad y_0{}^{1v} = -2x_0y_0{}^{111} - 6y_0{}^{11} = 0 - 6(0) = 0$

By Taylor's series formula is

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y^1(x_0) + \frac{(x-x_0)^2}{2!}y^{11}(x_0) + \frac{(x-x_0)^3}{3!}y^{111}(x_0) + \frac{(x-x_0)^4}{4!}y^{1V}(x_0) + \cdots$$

$$y(x) = y_0 + \frac{x}{1!}y_0{}^1 + \frac{x^2}{2!}y_0{}^{11} + \frac{x^3}{3!}y_0{}^{111} + \frac{x^4}{4!}y_0{}^{1V} + \cdots$$

$$y_1 = y(0.2) = 0 + (0.2) + \tfrac{(0.2)^2}{2}(0) + \tfrac{(0.2)^3}{6}(-4) + \tfrac{(0.2)^4}{24}(0) + \cdots = 0.2 - 0.005333 = 0.194667 \approx 0.1947$$

4. Evaluate $y(0.4)$ correct to four decimal places by Taylor's series method if $y^1 = x^2 + y^2$ $with\ y(0) = 0$

Sol) Given $y^1 = x^2 + y^2 \rightarrow (1)$

Differentiating (1) w.r.to 'x' successively, we get

$y^{11} = 2x + 2yy^1 \rightarrow (2)$, $y^{111} = 2 + 2yy^{11} + 2(y^1)^2 \rightarrow (3)$

$y^{1v} = 0 + 2yy^{111} + 2y^1y^{11} + 4y^1y^{11} = 2yy^{111} + 6y^1y^{11} \rightarrow (4)$ and so on.

We have $x_0 = 0, y_0 = 0$ substituting these values in (1),(2),(3),(4) we get

$y_0{}^1 = (x_0)^2 + (y_0)^2 = 0 + 0 = 0 \qquad y_0{}^{11} = 2x_0 + 2y_0y_0{}^1 = 0 + 0 = 0$

$y_0{}^{111} = 2 + 2y_0y_0{}^{11} + 2(y_0{}^1)^2 = 2 + 2(0) + 0 = 2 \qquad y_0{}^{1v} = 2y_0y_0{}^{111} + 6y_0{}^1y_0{}^{11} = 0 + 6(0) = 0$

By Taylor's series formula is

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y^1(x_0) + \frac{(x-x_0)^2}{2!}y^{11}(x_0) + \frac{(x-x_0)^3}{3!}y^{111}(x_0) + \frac{(x-x_0)^4}{4!}y^{1V}(x_0) + \cdots$$

$$y(x) = y_0 + \frac{x}{1!}y_0{}^1 + \frac{x^2}{2!}y_0{}^{11} + \frac{x^3}{3!}y_0{}^{111} + \frac{x^4}{4!}y_0{}^{1V} + \cdots$$

$$y_1 = y(0.4) = 0 + (0.4)(0) + \tfrac{(0.4)^2}{2}(0) + \tfrac{(0.4)^3}{6}(2) + \tfrac{(0.4)^4}{24}(0) + \cdots = \tfrac{0.064}{3} = 0.02133$$

5. Evaluate $y(0.1)$ and $y(0.2)$ correct to four decimal places by Taylor's series method if $y^1 = x^2 + y^2$ $with\ y(0) = 1$

Sol) Given $y^1 = x^2 + y^2 \rightarrow (1)$

Differentiating (1) w.r.to 'x' successively, we get

$y^{11} = 2x + 2yy^1 \rightarrow (2)$, $y^{111} = 2 + 2yy^{11} + 2(y^1)^2 \rightarrow (3)$

$y^{1v} = 0 + 2yy^{111} + 2y^1y^{11} + 4y^1y^{11} = 2yy^{111} + 6y^1y^{11} \rightarrow (4)$ and so on.

We have $x_0 = 0, y_0 = 1$ substituting these values in (1),(2),(3),(4) we get

$y_0{}^1 = (x_0)^2 + (y_0)^2 = 0 + 1 = 1 \qquad y_0{}^{11} = 2x_0 + 2y_0y_0{}^1 = 0 + 2(1)(1) = 2$

$y_0{}^{111} = 2 + 2y_0y_0{}^{11} + 2(y_0{}^1)^2 = 2 + 2(1)(2) + 2(1) = 8$ $\quad y_0{}^{1v} = 2y_0y_0{}^{111} + 6y_0{}^1y_0{}^{11} = 2(8) + 6(1)(2) = 28$

By Taylor's series formula is

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y^1(x_0) + \frac{(x-x_0)^2}{2!}y^{11}(x_0) + \frac{(x-x_0)^3}{3!}y^{111}(x_0) + \frac{(x-x_0)^4}{4!}y^{1V}(x_0) + \cdots$$

$$y(x) = y_0 + \frac{x}{1!}y_0{}^1 + \frac{x^2}{2!}y_0{}^{11} + \frac{x^3}{3!}y_0{}^{111} + \frac{x^4}{4!}y_0{}^{1V} + \cdots$$

$$y_1 = y(0.1) = 1 + (0.1)(1) + \tfrac{(0.1)^2}{2}(2) + \tfrac{(0.1)^3}{6}(8) + \tfrac{(0.1)^4}{24}(28) = 1 + 0.1 + 0.01 + 0.00133 + 0.0003417$$

$$= 1.1116717 \approx 1.1117$$

$$y_2 = y(0.2) = 1 + (0.2)(1) + \tfrac{(0.2)^2}{2}(2) + \tfrac{(0.2)^3}{6}(8) + \tfrac{(0.2)^4}{24}(28) = 1 + 0.2 + 0.04 + 0.01067 + 0.001867$$

$$=1.252537 \approx 1.2525$$

6. Find $y(0.1)$ correct to four decimal places by Taylor's series method if $y^1 = x - y^2, \quad y_0 = 1 \; when x_0 = 0$

Sol) Given $y^1 = x - y^2 \rightarrow (1)$

Differentiating (1) w.r.to 'x' successively, we get

$y^{11} = 1 - 2yy^1 \rightarrow (2), \quad y^{111} = 0 - 2yy^{11} - 2(y^1)^2 \rightarrow (3)$

$y^{1v} = -2yy^{111} - 2y^1y^{11} - 4y^1y^{11} = -2yy^{111} - 6y^1y^{11} \rightarrow (4)$ and so on.

We have $x_0 = 0, y_0 = 1$ substituting these values in (1),(2),(3),(4) we get

$y_0{}^1 = x_0 - (y_0)^2 = 0 - 1 = -1$ $\quad y_0{}^{11} = 1 - 2y_0y_0{}^1 = 1 - 2(1)(-1) = 3$

$y_0{}^{111} = -2y_0y_0{}^{11} - 2(y_0{}^1)^2 = -2(1)(3) - 2(-1)^2 = -8$ $\quad y_0{}^{1v} = -2y_0y_0{}^{111} - 6y_0{}^1y_0{}^{11} = -2(-8) - 6(-1)(3) = 34$

By Taylor's series formula is

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y^1(x_0) + \frac{(x-x_0)^2}{2!}y^{11}(x_0) + \frac{(x-x_0)^3}{3!}y^{111}(x_0) + \frac{(x-x_0)^4}{4!}y^{1V}(x_0) + \cdots$$

$$y(x) = y_0 + \frac{x}{1!}y_0{}^1 + \frac{x^2}{2!}y_0{}^{11} + \frac{x^3}{3!}y_0{}^{111} + \frac{x^4}{4!}y_0{}^{1V} + \cdots$$

$$y(0.1) = 1 + (0.1)(-1) + \tfrac{(0.1)^2}{2}(3) + \tfrac{(0.1)^3}{6}(-8) + \tfrac{(0.1)^4}{24}(34) = 1 - 0.1 + 0.015 - 0.00133 + 0.0001417$$

$$=0.9138117 \approx 0.9138$$

7. Find $y(0.2)$ correct to four decimal places by Taylor's series method if $y^1 = x + y^2$, $y_0 = 1\ when x_0 = 0$

Sol) Given $y^1 = x + y^2 \rightarrow (1)$

Differentiating (1) w.r.to 'x' successively, we get

$y^{11} = 1+2yy^1 \rightarrow (2)$, $y^{111} = 0 + 2yy^{11} + 2(y^1)^2 \rightarrow (3)$

$y^{1v} = 2yy^{111} + 2y^1y^{11} + 4y^1y^{11} = 2yy^{111} + 6y^1y^{11} \rightarrow (4)$ and so on.

We have $x_0 = 0, y_0 = 1$ substituting these values in (1),(2),(3),(4) we get

$y_0{}^1 = x_0 + (y_0)^2 = 0 + 1 = 1$ $\qquad$ $y_0{}^{11} = 1 + 2y_0y_0{}^1 = 1 + 2(1)(1) = 3$

$y_0{}^{111} = 2y_0y_0{}^{11} + 2(y_0{}^1)^2 = 2(1)(3) + 2(1)^2 = 8$ $\quad$ $y_0{}^{1v} = 2y_0y_0{}^{111} + 6y_0{}^1y_0{}^{11} = 2(8) + 6(1)(3) = 34$

By Taylor's series formula is

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!}y^1(x_0) + \frac{(x - x_0)^2}{2!}y^{11}(x_0) + \frac{(x - x_0)^3}{3!}y^{111}(x_0) + \frac{(x - x_0)^4}{4!}y^{1V}(x_0) + \cdots$$

$$y(x) = y_0 + \frac{x}{1!}y_0{}^1 + \frac{x^2}{2!}y_0{}^{11} + \frac{x^3}{3!}y_0{}^{111} + \frac{x^4}{4!}y_0{}^{1V} + \cdots$$

$$y(0.2) = 1 + (0.2)(1) + \frac{(0.2)^2}{2}(3) + \frac{(0.2)^3}{6}(8) + \frac{(0.2)^4}{24}(34) = 1 + 0.2 + 0.06 + 0.01067 + 0.002267$$

$=1.272937 \approx 1.2729$

8. Find $y(0.2)$ correct to four decimal places by Taylor's series method if $y^1 = x + y^2$, $y_0 = 0\ when x_0 = 0$

Sol) Given $y^1 = x + y^2 \rightarrow (1)$

Differentiating (1) w.r.to 'x' successively, we get

$y^{11} = 1+2yy^1 \rightarrow (2)$, $y^{111} = 0 + 2yy^{11} + 2(y^1)^2 \rightarrow (3)$

$y^{1v} = 2yy^{111} + 2y^1y^{11} + 4y^1y^{11} = 2yy^{111} + 6y^1y^{11} \rightarrow (4)$ and so on.

We have $x_0 = 0, y_0 = 0$ substituting these values in (1),(2),(3),(4) we get

$y_0{}^1 = x_0 + (y_0)^2 = 0 + 0 = 0$ $\qquad$ $y_0{}^{11} = 1 + 2y_0y_0{}^1 = 1 + 2(0)(0) = 1$

$y_0{}^{111} = 2y_0y_0{}^{11} + 2(y_0{}^1)^2 = 2(0)(1) + 2(0)^2 = 0,$ $\quad$ $y_0{}^{1v} = 2y_0y_0{}^{111} + 6y_0{}^1y_0{}^{11} = 2(0) + 6(0)(1) = 0$

By Taylor's series formula is

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!}y^1(x_0) + \frac{(x - x_0)^2}{2!}y^{11}(x_0) + \frac{(x - x_0)^3}{3!}y^{111}(x_0) + \frac{(x - x_0)^4}{4!}y^{1V}(x_0) + \cdots$$

$$y(x) = y_0 + \frac{x}{1!}y_0{}^1 + \frac{x^2}{2!}y_0{}^{11} + \frac{x^3}{3!}y_0{}^{111} + \frac{x^4}{4!}y_0{}^{1V} + \cdots$$

$y(0.2) = 0 + (0.2)(0) + \frac{(0.2)^2}{2}(1) + \frac{(0.2)^3}{6}(0) + \frac{(0.2)^4}{24}(0) = 0 + 0 + 0.02 + 0 + 0 = 0.02$

9. Find $y(0.1)$ correct to four decimal places by Taylor's series method if $y^1 = x + y, \quad y_0 = 1\ when x_0 = 0$

Sol) Given $y^1 = \ x + y \ \rightarrow (1)$

Differentiating (1) w.r.to 'x' successively, we get

$y^{11} = 1{+}y^1 \rightarrow (2), \quad y^{111} = 0 + y^{11} \rightarrow (3)$

$y^{1v} = y^{111} \rightarrow (4)$ and so on.

We have $x_0 = 0, y_0 = 1$ substituting these values in (1),(2),(3),(4) we get

$y_0{}^1 = x_0 + y_0 = 0 + 1 = 1 \qquad\qquad y_0{}^{11} = 1 + y_0{}^1 = 1 + 1 = 2$

$y_0{}^{111} = \ y_0{}^{11} = 2 \quad y_0{}^{1v} = \ y_0{}^{111} = 2$

By Taylor's series formula is

$$y(x) = \ y(x_0) + \frac{(x - x_0)}{1!}y^1(x_0) + \frac{(x - x_0)^2}{2!}y^{11}(x_0) + \frac{(x - x_0)^3}{3!}y^{111}(x_0) + \frac{(x - x_0)^4}{4!}y^{1V}(x_0) + \cdots$$

$$y(x) = y_0 + \frac{x}{1!}y_0{}^1 + \frac{x^2}{2!}y_0{}^{11} + \frac{x^3}{3!}y_0{}^{111} + \frac{x^4}{4!}y_0{}^{1V} + \cdots$$

$y(0.1) = 1 + (0.1)(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(2) + \frac{(0.1)^4}{24}(2) + \cdots = 1 + 0.1 + 0.01 + 0.000333 + 0.0000083$

$= 1.1103413 \approx 1.1103$

9. Find $y(0.2)$ correct to four decimal places by Taylor's series method if $y^1 = 1 - y, \quad y_0 = 0\ when x_0 = 0$

Sol) Given $y^1 = \ 1 - y \ \rightarrow (1)$

Differentiating (1) w.r.to 'x' successively, we get

$y^{11} = -y^1 \rightarrow (2), \quad y^{111} = -y^{11} \rightarrow (3)$

$y^{1v} = -y^{111} \rightarrow (4)$ and so on.

We have $x_0 = 0, y_0 = 0$ substituting these values in (1),(2),(3),(4) we get

$y_0{}^1 = 1 - y_0 = 1 - 0 = 1 \qquad\qquad y_0{}^{11} = -y_0{}^1 = -1$

$y_0{}^{111} = -\ y_0{}^{11} = -1 \quad y_0{}^{1v} = \ -y_0{}^{111} = -1$

By Taylor's series formula is

$$y(x) = \ y(x_0) + \frac{(x - x_0)}{1!}y^1(x_0) + \frac{(x - x_0)^2}{2!}y^{11}(x_0) + \frac{(x - x_0)^3}{3!}y^{111}(x_0) + \frac{(x - x_0)^4}{4!}y^{1V}(x_0) + \cdots$$

$$y(x) = y_0 + \frac{x}{1!}y_0{}^1 + \frac{x^2}{2!}y_0{}^{11} + \frac{x^3}{3!}y_0{}^{111} + \frac{x^4}{4!}y_0{}^{1V} + \cdots$$

$$y(0.2) = 0 + (0.2)(1) + \tfrac{(0.2)^2}{2}(-1) + \tfrac{(0.2)^3}{6}(-1) + \tfrac{(0.2)^4}{24}(-1) + \cdots = 0 + 0.2 - 0.02 - 0.00133 - 0.000067$$

$$= 0.178603 \approx 0.1786$$

10. Using Taylor's series method, find the solution of the initial value problem $y^1 = x + y,$ $y(1) = 0 \;\; at x = 1.2, with\ h = 0.1$ and compare the result with the exact solution.

Sol) Given $y^1 = \;\; x + y \;\; \to (1)$

Differentiating (1) w.r.to 'x' successively, we get

$y^{11} = 1{+}y^1 \to (2), \;\; y^{111} = 0 + y^{11} \to (3)$

$y^{1v} = y^{111} \to (4)$ and so on.

We have $x_0 = 1, y_0 = 0$ substituting these values in (1),(2),(3),(4) we get

$y_0{}^1 = x_0 + y_0 = 1 + 0 = 1$ $\qquad y_0{}^{11} = 1 + y_0{}^1 = 1 + 1 = 2$

$y_0{}^{111} = \; y_0{}^{11} = 2 \;\;\; y_0{}^{1v} = \; y_0{}^{111} = 2$

By Taylor's series formula is

$$y(x) = \; y(x_0) + \frac{(x - x_0)}{1!}y^1(x_0) + \frac{(x - x_0)^2}{2!}y^{11}(x_0) + \frac{(x - x_0)^3}{3!}y^{111}(x_0) + \frac{(x - x_0)^4}{4!}y^{1V}(x_0) + \cdots$$

$$y(x) = y_0 + \tfrac{(x-1)}{1!}y_0{}^1 + \tfrac{(x-1)^2}{2!}y_0{}^{11} + \tfrac{(x-1)^3}{3!}y_0{}^{111} + \tfrac{(x-1)^4}{4!}y_0{}^{1V} + \cdots$$

$$y(1.1) = 1 + \frac{(1.1 - 1)}{1}(1) + \frac{(1.1 - 1)^2}{2}(2) + \frac{(1.1 - 1)^3}{6}(2) + \frac{(1.1 - 1)^4}{24}(2) + \cdots$$

$$y(0.1) = 0 + (0.1)(1) + \tfrac{(0.1)^2}{2}(2) + \tfrac{(0.1)^3}{6}(2) + \tfrac{(0.1)^4}{24}(2) + \cdots = 0 + 0.1 + 0.01 + 0.000333 + 0.0000083$$

$$= 0.1103413 \approx 0.1103$$

$$y(0.2) = 0 + (0.2)(1) + \tfrac{(0.2)^2}{2}(2) + \tfrac{(0.2)^3}{6}(2) + \tfrac{(0.2)^4}{24}(2) + \cdots = 0.2 + 0.04 + 0.002667 + 0.0001333$$

$$= 0.2428003 \approx 0.2428$$

Euler's method:

1. Explain the method of solving the differential equation $\frac{dy}{dx} = f(x,y)\ with\ y(x_0) = y_0 by\ Euler's\ method$

2. Solve $y^1 = x + y\ , y(0) = 0$ choose h=0.2 and compute y (0.4) y(0.6) by Euler's method

Sol) Given $y^1 = x + y = f(x,y)\ x_0 = 0, y_0 = 0\ and\ \ h = 0.2$

x		y	
x_0	0	0	y_0
x_1	0.2	0	y_1
x_2	0.4	0.04	y_2
x_3	0.6	0.128	y_3

The Euler formula is $y_{n+1} = y_n + hf(x_n, y_n) \longrightarrow (1) \quad where\ n = 0,1,2, \ldots$

$put\ n = 0\ in\ (1),\ y_1 = y_0 + hf(x_0, y_0) \Rightarrow y_1 = 0 + 0.2(x_0 + y_0) = 0$

$n = 1\ in\ (1), y_2 = y_1 + hf(x_1, y_1) \Rightarrow y_2 = 0 + 0.2(x_1 + y_1) = 0.2(0.2 + 0) = 0.04$

$n = 2\ in\ (1), y_3 = y_2 + hf(x_2, y_2) \Rightarrow y_3 = 0.04 + 0.2(x_2 + y_2) = 0.04 + 0.2(0.4 + 0.04) = 0.128$

Hence , $y_2 = y(0.4) = 0.04\ and\ y_3 = 0.128$

3. Solve $y^1 = x + y\ , y(0) = 1$ and compute y(0.05) y(0.1) by Euler's method

Sol) Given $y^1 = x + y = f(x,y)\ x_0 = 0, y_0 = 1\ and\ take\ h = 0.05$

x		y	
x_0	0	0	y_0
x_1	0.05	1	y_1
x_2	0.1	1.0525	y_2

The Euler formula is $y_{n+1} = y_n + hf(x_n, y_n) \longrightarrow (1) \quad where\ n = 0,1,2, \ldots$

$put\ n = 0\ in\ (1),\ y_1 = y_0 + hf(x_0, y_0) \Rightarrow y_1 = 1 + 0.05(x_0 + y_0)$

$= 1 + 0.05(0 + 0) = 1$

$n = 1\ in\ (1), y_2 = y_1 + hf(x_1, y_1) \Rightarrow y_2 = 1 + 0.05(x_1 + y_1) = 1 + 0.05(0.05 + 1) = 1.0525$

Hence , $y_1 = y(0.05) = 1\ and\ y_2 = y(0.1) = 1.0525$

4. Solve $y^1 = x + y\ , y(0) = 1$ choose h=0.05 and compute y(0.3) by Euler's method

Sol) Given $y^1 = x + y = f(x,y)\ x_0 = 0, y_0 = 1\ and\ \ h = 0.05$

x		y	
x_0	0	1	y_0
x_1	0.05	1.05	y_1
x_2	0.1	1.1075	y_2
x_3	0.15	1.1704	y_3
x_4	0.2	1.2391	y_4
x_5	0.25	1.3136	y_5
x_6	0.3	1.3918	y_6

The Euler formula is $y_{n+1} = y_n + hf(x_n, y_n) \longrightarrow (1) \quad where\ n = 0,1,2, \ldots$

$put\ n = 0\ in\ (1),\ y_1 = y_0 + hf(x_0, y_0) \Rightarrow y_1 = 1 + 0.05(x_0 + y_0)$

$= 1 + 0.05(0 + 1) = 1.05$

$n = 1\ in\ (1), y_2 = y_1 + hf(x_1, y_1) \Rightarrow y_2 = 1.05 + 0.05(x_1 + y_1)$

$= 1.05 + 0.05(0.1 + 1.05) = 1.1075$

$n = 2\ in\ (1), y_3 = y_2 + hf(x_2, y_2) \Rightarrow y_3 = 1.1075 + 0.05(x_2 + y_2) = 1.1075 + 0.05(0.15 + 1.1075)$
$= 1.1704$

$n = 3\ in\ (1), y_4 = y_3 + hf(x_3, y_3) \Rightarrow y_4 = 1.1704 + 0.05(x_3 + y_3) = 1.1704 + 0.05(0.2 + 1.1704)$
$= 1.2391$

$n = 4\ in\ (1), y_5 = y_4 + hf(x_4, y_4) \Rightarrow y_5 = 1.2391 + 0.05(x_4 + y_4) = 1.2391 + 0.05(0.25 + 1.2391)$
$= 1.3136$

$n = 5\ in\ (1), y_6 = y_5 + hf(x_5, y_5) \Rightarrow y_6 = 1.3136 + 0.05(x_5 + y_5) = 1.3136 + 0.05(0.25 + 1.3136)$
$= 1.3918$

Hence $y_6 = y(0.3) = 1.3918$

5. Solve $y^1 = x + y$, $y(0) = 1$ choose h=0.1 and compute y(0.3) by Euler's method

Sol) Given $y^1 = x + y = f(x, y)$ $x_0 = 0, y_0 = 1$ *and* $h = 0.1$

x		y	
x_0	0	1	y_0
x_1	0.1	1.1	y_1
x_2	0.2	1.22	y_2
x_3	0.3	1.362	y_3

The Euler formula is $y_{n+1} = y_n + hf(x_n, y_n) \longrightarrow (1) \quad where\ n = 0,1,2,\ldots$

$put\ n = 0\ in\ (1),\ y_1 = y_0 + hf(x_0, y_0) \Rightarrow y_1 = 1 + 0.1(x_0 + y_0)$

$= 1 + 0.1(0 + 1) = 1.1$

$n = 1\ in\ (1), y_2 = y_1 + hf(x_1, y_1) \Rightarrow y_2 = 1.1 + 0.1(x_1 + y_1) = 1.1 + 0.1(0.1 + 1.1) = 1.22$

$n = 2\ in\ (1), y_3 = y_2 + hf(x_2, y_2) \Rightarrow y_3 = 1.22 + 0.1(x_2 + y_2) = 1.22 + 0.1(0.2 + 1.22) = 1.362$

Hence $y_3 = y(0.3) = 1.362$

6. Solve $y^1 = x + y$, $y(0) = 1$ choose h=0.1 and compute y(0.5) by Euler's method

Sol) Given $y^1 = x + y = f(x, y)$ $x_0 = 0, y_0 = 1$ *and* $h = 0.1$

x		y	
x_0	0	1	y_0
x_1	0.1	1.1	y_1
x_2	0.2	1.22	y_2
x_3	0.3	1.362	y_3
x_4	0.4	1.5282	y_4
x_5	0.5	1.7110	y_5

The Euler formula is $y_{n+1} = y_n + hf(x_n, y_n) \longrightarrow (1) \quad where\ n = 0,1,2,\ldots$

$put\ n = 0\ in\ (1),\ y_1 = y_0 + hf(x_0, y_0) \Rightarrow y_1 = 1 + 0.1(x_0 + y_0)$

$= 1 + 0.1(0 + 1) = 1.1$

$n = 1\ in\ (1), y_2 = y_1 + hf(x_1, y_1) \Rightarrow y_2 = 1.1 + 0.1(x_1 + y_1)$

$= 1.1 + 0.1(0.1 + 1.1) = 1.22$

$n = 2\ in\ (1), y_3 = y_2 + hf(x_2, y_2) \Rightarrow y_3 = 1.22 + 0.1(x_2 + y_2) = 1.22 + 0.1(0.2 + 1.22) = 1.362$

$n = 3\ in\ (1), y_4 = y_3 + hf(x_3, y_3) \Rightarrow y_4 = 1.362 + 0.1(x_3 + y_3) = 1.362 + 0.1(0.3 + 1.362) = 1.5282$

$n = 4\ in\ (1), y_5 = y_4 + hf(x_4, y_4) \Rightarrow y_5 = 1.5282 + 0.1(x_4 + y_4) = 1.5282 + 0.1(0.3 + 1.5282) = 1.7110$

Hence $y_5 = y(0.5) = 1.7110$

7. Solve $y^1 = x + y + xy, \quad y(0) = 1$ choose h=0.025 and compute y(0.1) by Euler's method

Sol) Given $y^1 = x + y + xy = f(x, y)$ $x_0 = 0, y_0 = 1$ *and* $h = 0.025$

The Euler formula is $y_{n+1} = y_n + hf(x_n, y_n) \longrightarrow (1) \quad where\ n = 0,1,2,\ldots$

$put\ n = 0\ in\ (1),\ y_1 = y_0 + hf(x_0, y_0) \Rightarrow y_1 = 1 + 0.025(x_0 + y_0 + x_0y_0)$

$= 1 + 0.025(0 + 1 + 0) = 1.025$

$n = 1\ in\ (1), y_2 = y_1 + hf(x_1, y_1) \Rightarrow y_2 = 1.025 + 0.025(x_1 + y_1 + x_1y_1)$

$= 1.025 + 0.025(0.025 + 1.025 + 0.025(1.025) = 1.0519$

	x	y	
x_0	0	1	y_0
x_1	0.025	1.025	y_1
x_2	0.05	1.0519	y_2
x_3	0.075	1.0808	y_3
x_4	0.1	1.1117	y_4

$n = 2\ in\ (1), y_3 = y_2 + hf(x_2, y_2) \Rightarrow y_3 = 1.0519 + 0.025(x_2 + y_2 + x_2y_2) = 1.0808$

$n = 3\ in\ (1), y_4 = y_3 + hf(x_3, y_3) \Rightarrow y_4 = 1.0808 + 0.025(x_3 + y_3 + x_3y_3) = 1.1117$

Hence $y_4 = y(0.1) = 1.1117$

8. Solve $y^1 = x + y + xy$, $y(0) = 1$ choose h=0.05 and compute y(0.3) by Euler's method

9. Solve $y^1 = \frac{y-x}{y+x}$ $y(0) = 1$ and compute y at x=0.1 in five steps by Euler's method

Sol) Given $f(x, y) = \frac{y-x}{y+x}$ $x_0 = 0, y_0 = 1$ number of steps is 5 so $h = \frac{0.1}{5} = 0.02$

The Euler formula is $y_{n+1} = y_n + hf(x_n, y_n) \longrightarrow (1)$ $where\ n = 0,1,2, \dots$

$put\ n = 0\ in\ (1),\ y_1 = y_0 + hf(x_0, y_0) \Rightarrow y_1 = 1 + 0.02\left[\frac{y_0 - x_0}{y_0 + x_0}\right]$

$= 1 + 0.02\left(\frac{1-0}{1+0}\right) = 1.02$

$n = 1\ in\ (1), y_2 = y_1 + hf(x_1, y_1) \Rightarrow y_2 = 1.02 + 0.02\left(\frac{1.02-0.02}{1.02+0.02}\right) = 1.0392$

	x	y	
x_0	0	1	y_0
x_1	0.02	1.02	y_1
x_2	0.04	1.0392	y_2
x_3	0.06	1.0577	y_3
x_4	0.08	1.0738	y_4
x_5	0.1	1.0910	y_5

$n = 2\ in\ (1), y_3 = y_2 + hf(x_2, y_2) \Rightarrow y_3 = 1.0392 + 0.02\left(\frac{1.0392 - 0.04}{1.0392 + 0.04}\right) = 1.0577$

$n = 3\ in\ (1), y_4 = y_3 + hf(x_3, y_3) \Rightarrow y_4 = 1.0577 + 0.02\left(\frac{1.0577 - 0.06}{1.0577 + 0.06}\right) = 1.0738$

$n = 4\ in\ (1), y_5 = y_4 + hf(x_4, y_4) \Rightarrow y_5 = 1.0738 + 0.02\left(\frac{1.0738 - 0.08}{1.0738 + 0.08}\right) = 1.0910$

Hence $y_5 = y(0.1) = 1.0910$

10. Solve $y^1 = \frac{y-x}{y+x}$ $y(0) = 1$ and compute y at x=0.1 in 4 steps by Euler's method. Ans: y(0.1)=1.0932

11. Solve $y^1 = \frac{y-x}{y+x}$ $y(0) = 1$ and compute y at x=0.6 and taking h=0.2 by Euler's method

Sol) Given $f(x, y) = \frac{y-x}{y+x}$ $x_0 = 0, y_0 = 1$ and $h = 0.2$

x		y	
x_0	0	1	y_0
x_1	0.2	1.2	y_1
x_2	0.4	1.3429	y_2
x_3	0.6	1.4508	y_3

The Euler formula is $y_{n+1} = y_n + hf(x_n, y_n) \longrightarrow (1) \quad where\ n = 0,1,2,\ldots$

$put\ n = 0\ in\ (1),\ y_1 = y_0 + hf(x_0, y_0) \Rightarrow y_1 = 1 + 0.2\left[\frac{y_0 - x_0}{y_0 + x_0}\right] = 1 + 0.02\left(\frac{1-0}{1+0}\right) = 1.2$

$n = 1\ in\ (1), y_2 = y_1 + hf(x_1, y_1) \Rightarrow y_2 = 1.2 + 0.2\left(\frac{1.2-0.2}{1.2+0.2}\right) = 1.3429$

$n = 2\ in\ (1), y_3 = y_2 + hf(x_2, y_2) \Rightarrow y_3 = 1.3429 + 0.2\left(\frac{1.3429 - 0.4}{1.3429 + 0.4}\right) = 1.4508$

Hence $y_3 = y(0.6) = 1.4508$

12. Solve $y^1 = x + y^2$, $y(0) = 1$ and compute y(0.1 to 0.5) by Euler's method

Sol) Given $y^1 = x + y = f(x, y)\ x_0 = 0, y_0 = 1\ and\ h = 0.1$

x		y	
x_0	0	1	y_0
x_1	0.1	1.1	y_1
x_2	0.2	1.231	y_2
x_3	0.3	1.4025	y_3
x_4	0.4	1.6292	y_4
x_5	0.5	1.9346	y_5

The Euler formula is $y_{n+1} = y_n + hf(x_n, y_n) \longrightarrow (1) \quad where\ n = 0,1,2,\ldots$

$put\ n = 0\ in\ (1),\ y_1 = y_0 + hf(x_0, y_0) \Rightarrow y_1 = 1 + 0.1(x_0 + {y_0}^2)$

$= 1 + 0.1(0 + (1)^2) = 1.1$

$n = 1\ in\ (1), y_2 = y_1 + hf(x_1, y_1) \Rightarrow y_2 = 1.1 + 0.1(x_1 + {y_1}^2)$

$= 1.1 + 0.1(0.1 + (1.1)^2) = 1.231$

$n = 2\ in\ (1), y_3 = y_2 + hf(x_2, y_2) \Rightarrow y_3 = 1.231 + 0.1(x_2 + {y_2}^2) = 1.231 + 0.1(0.2 + (1.231)^2) = 1.4025$

$n = 3\ in\ (1), y_4 = y_3 + hf(x_3, y_3) \Rightarrow y_4 = 1.4025 + 0.1(x_3 + {y_3}^2) = 1.4025 + 0.1(0.3 + (1.4025)^2)$
$= 1.6292$

$n = 4\ in\ (1), y_5 = y_4 + hf(x_4, y_4) \Rightarrow y_5 = 1.6292 + 0.1(x_4 + {y_4}^2) = 1.6292 + 0.1(0.4 + (1.6292)^2)$
$= 1.9346$

Hence $y_5 = y(0.5) = 1.9346$

13. Solve $y^1 = 1 + y^2 \quad y(0) = 0$ and compute y at x=0.3 by Euler's method

Sol) Given $f(x, y) = 1 + y^2 \quad x_0 = 0, y_0 = 0$ and $h = 0.1$

The Euler formula is $y_{n+1} = y_n + hf(x_n, y_n) \longrightarrow (1) \quad where\ n = 0,1,2,\ldots$

x		y	
x_0	0	0	y_0
x_1	0.1	0.1	y_1
x_2	0.2	0.201	y_2
x_3	0.3	0.3050	y_3

$put\ n = 0\ in\ (1),\ y_1 = y_0 + hf(x_0, y_0) \Rightarrow y_1 = 0 + 0.1(1 + {y_0}^2) = 0.1(1 + 0) = 0.1$

$n = 1\ in\ (1), y_2 = y_1 + hf(x_1, y_1) \Rightarrow y_2 = 0.1 + 0.1(1 + (0.1)^2) = 0.201$

$n = 2\ in\ (1), y_3 = y_2 + hf(x_2, y_2) \Rightarrow y_3 = 0.201 + 0.1(1 + (0.201)^2 = 0.3050$

Hence $y_3 = y(0.3) = 0.3050$

14. Solve $y^1 = -y \quad y(0) = 1$ and compute y at x=0.2 by Euler's method

Sol) Given $f(x,y) = -y \quad x_0 = 0, y_0 = 1$ and $h = 0.1$

The Euler formula is $y_{n+1} = y_n + hf(x_n, y_n) \longrightarrow (1) \quad where\ n = 0,1,2,\ldots$

$put\ n = 0\ in\ (1),\ y_1 = y_0 + hf(x_0, y_0) \Rightarrow y_1 = 1 + 0.1(-y_0) = 1 + 0.1(-1) = 0.9$

$n = 1\ in\ (1), y_2 = y_1 + hf(x_1, y_1) \Rightarrow y_2 = 0.9 + 0.1(-0.9) = 0.81$

Hence $y_2 = y(0.2) = 0.81$

x		y	
x_0	0	1	y_0
x_1	0.1	0.9	y_1
x_2	0.2	0.81	y_2

15. Solve $y^1 = 1 - y \quad y(0) = 0$ and compute y at x=0.3 by Euler's method

Sol) Given $f(x,y) = 1 - y \quad x_0 = 0, y_0 = 0$ and $h = 0.1$

The Euler formula is $y_{n+1} = y_n + hf(x_n, y_n) \longrightarrow (1) \quad where\ n = 0,1,2,\ldots$

$put\ n = 0\ in\ (1),\ y_1 = y_0 + hf(x_0, y_0) \Rightarrow y_1 = 0 + 0.1(1 - y_0) = 0.1(1 - 0) = 0.1$

$n = 1\ in\ (1), y_2 = y_1 + hf(x_1, y_1) \Rightarrow y_2 = 0.1 + 0.1(1 - 0.1) = 0.19$

$n = 2\ in\ (1), y_3 = y_2 + hf(x_2, y_2) \Rightarrow y_3 = 0.19 + 0.1(1 - 0.19) = 0.271$

Hence $y_3 = y(0.3) = 0.271$

x		y	
x_0	0	0	y_0
x_1	0.1	0.1	y_1
x_2	0.2	0.19	y_2
x_3	0.3	0.271	y_3

16. Solve $y^1 + 2y = 0 \quad y(0) = 1$ and compute y at x=0.3 by Euler's method

Sol) Given $f(x,y) = -2y \quad x_0 = 0, y_0 = 1$ and $h = 0.1$

The Euler formula is $y_{n+1} = y_n + hf(x_n, y_n) \longrightarrow (1)$

$where\ n = 0,1,2,\ldots$

x		y	
x_0	0	1	y_0
x_1	0.1	0.8	y_1
x_2	0.2	0.64	y_2
x_3	0.3	0.512	y_3

$put\ n = 0\ in\ (1),\ y_1 = y_0 + hf(x_0, y_0) \Rightarrow y_1 = 1 + 0.1(-2y_0) = 1 + 0.1(-2) = 0.8$

$n = 1\ in\ (1), y_2 = y_1 + hf(x_1, y_1) \Rightarrow y_2 = 0.8 + 0.1(-2(0.8)) = 0.64$

$n = 2\ in\ (1), y_3 = y_2 + hf(x_2, y_2) \Rightarrow y_3 = 0.64 + 0.1(-2)(0.64) = 0.512$

Hence $y_3 = y(0.3) = 0.512$

17. Solve $y^1 = x^2 - y, y(0) = 1$ and compute y(0.4) by Euler's method

Sol) Given $y^1 = x^2 - y = f(x,y)\ x_0 = 0, y_0 = 1\ and\ \ h = 0.1$

The Euler formula is $y_{n+1} = y_n + hf(x_n, y_n) \longrightarrow (1)\ where\ n = 0,1,2,\ldots$

$put\ n = 0\ in\ (1),\ y_1 = y_0 + hf(x_0, y_0) \Rightarrow y_1 = 1 + 0.1({x_0}^2 - y_0)$

$= 1 + 0.1(0 - 1) = 0.9$

x		y	
x_0	0	1	y_0
x_1	0.1	0.9	y_1
x_2	0.2	0.811	y_2
x_3	0.3	0.7339	y_3
x_4	0.4	0.6695	y_4

$n = 1\ in\ (1), y_2 = y_1 + hf(x_1, y_1) \Rightarrow y_2 = 0.9 + 0.1({x_1}^2 - y_1)$

$= 0.9 + 0.1((0.1)^2 - 0.9) = 0.811$

$n = 2\ in\ (1), y_3 = y_2 + hf(x_2, y_2) \Rightarrow y_3 = 0.811 + 0.1({x_2}^2 - y_2) = 0.811 + 0.1((0.2)^2 - 0.811) = 0.7339$

$n = 3\ in\ (1), y_4 = y_3 + hf(x_3, y_3) \Rightarrow y_4 = 0.7339 + 0.1({x_3}^2 - y_3) = 0.7339 + 0.1((0.3)^2 - 0.7339)$
$= 0.6695$

Hence $y_4 = y(0.4) = 0.6695$

18. Solve $y^1 = 1 - 2xy\ , y(0) = 0$ and compute y(0.4) by Euler's method

x		y	
x_0	0	0	y_0
x_1	0.1	0.1	y_1
x_2	0.2	0.198	y_2
x_3	0.3	0.2901	y_3
x_4	0.4	0.3727	y_4

Sol) Given $y^1 = 1 - 2xy = f(x, y)\ x_0 = 0, y_0 = 0\ and\ take\ h = 0.1$

The Euler formula is $y_{n+1} = y_n + hf(x_n, y_n) \longrightarrow (1) \quad where\ n = 0,1,2, \ldots$

$put\ n = 0\ in\ (1),\ y_1 = y_0 + hf(x_0, y_0) \Rightarrow y_1 = 0 + 0.1(1 - 2x_0y_0)$

$= 0 + 0.1(1 - 2(0)) = 0.1$

$n = 1\ in\ (1), y_2 = y_1 + hf(x_1, y_1) \Rightarrow y_2 = 0.1 + 0.1(1 - 2x_1y_1)$

$= 0.1 + 0.1(1 - 2(0.1)(0.1)) = 0.198$

$n = 2\ in\ (1), y_3 = y_2 + hf(x_2, y_2) \Rightarrow y_3 = 0.198 + 0.1(1 - 2x_2y_2) = 0.198 + 0.1(1 - 2(0.2)(0.198))$
$= 0.2901$

$n = 3\ in\ (1), y_4 = y_3 + hf(x_3, y_3) \Rightarrow y_4 = 0.2901 + 0.1(1 - 2x_3y_3) = 0.2901 + 0.1(1 - 2(0.3)(0.2901))$
$= 0.3727$

Hence $y_4 = y(0.4) = 0.3727$

19. Solve $y^1 = 1 - 2xy\ , y(0) = 0$ and compute y(0.6) taking h=0.2 by Euler's method

Sol) Given $y^1 = 1 - 2xy = f(x, y)\ x_0 = 0, y_0 = 0\ and\ take\ h = 0.2$

x		y	
x_0	0	0	y_0
x_1	0.2	0.2	y_1
x_2	0.4	0.384	y_2
x_3	0.6	0.5226	y_3

The Euler formula is $y_{n+1} = y_n + hf(x_n, y_n) \longrightarrow (1)\ where\ n = 0,1,2, \ldots$

$put\ n = 0\ in\ (1),\ y_1 = y_0 + hf(x_0, y_0) \Rightarrow y_1 = 0 + 0.2(1 - 2x_0y_0)$

$= 0 + 0.2(1 - 2(0)) = 0.2$

$n = 1\ in\ (1), y_2 = y_1 + hf(x_1, y_1) \Rightarrow y_2 = 0.2 + 0.2(1 - 2x_1y_1)$

$= 0.2 + 0.2(1 - 2(0.2)(0.2)) = 0.384$

$n = 2\ in\ (1), y_3 = y_2 + hf(x_2, y_2) \Rightarrow y_3 = 0.384 + 0.2(1 - 2x_2y_2) = 0.384 + 0.2\big(1 - 2(0.4)(0.384)\big)$

$= 0.5226$

Hence $y_3 = y(0.6) = 0.5226$

20. Solve $y^1 = -y \quad y(0) = 1$ and compute y at x=0.04 with h=0.01 by Euler's method. compare the result with exact solution.

	x	y	
x_0	0	1	y_0
x_1	0.01	0.99	y_1
x_2	0.02	0.9801	y_2
x_3	0.03	0.9703	y_3
x_4	0.04	0.9606	y_4

Sol) Given $f(x,y) = -y \quad x_0 = 0, y_0 = 1$ and $h = 0.01$

The Euler formula is $y_{n+1} = y_n + hf(x_n, y_n) \longrightarrow (1) \quad where\ n = 0,1,2,\ldots$

$put\ n = 0\ in\ (1),\ y_1 = y_0 + hf(x_0, y_0) \Rightarrow y_1 = 1 + 0.01(-y_0) = 1 + 0.01(-1)$

$= 0.99$

$n = 1\ in\ (1), y_2 = y_1 + hf(x_1, y_1) \Rightarrow y_2 = 0.99 + 0.01(-0.99) = 0.9801$

$n = 2\ in\ (1), y_3 = y_2 + hf(x_2, y_2) \Rightarrow y_3 = 0.9801 + 0.01(-0.9801) = 0.9703$

$n = 1\ in\ (1), y_4 = y_3 + hf(x_3, y_3) \Rightarrow y_4 = 0.9703 + 0.01(-0.9703) = 0.9606$

Hence $y_4 = y(0.04) = 0.9606 \to (1)$

Exact solution: Given that $y^1 = -y \quad y(0) = 1$

$$y^1 = -y \ \Rightarrow \frac{dy}{y} = -dx \ \Rightarrow \int \frac{dy}{y} = -\int dx \Rightarrow logy = -x + c \Rightarrow y = e^{-x+c} = ke^{-x} \to (2)$$

Initial condition is $y = 1\ when\ x = 0$

From (1), we have k=1

Equation (1) becomes $y = e^{-x} \Rightarrow y(0.04) = e^{-0.04} = 0.9608 \to (3)$

Comparing approximate result (1) and exact solution (3), we observe that they agree up to 3 decimals.

Modified Euler's method:

1. Explain the method of solving the differential equation $\frac{dy}{dx} = f(x,y)\ with\ y(x_0) = y_0 by$ modified Euler's method.

2. Solve $y^1 = x^2 + y\ with\ y(0) = 1$ and find y (0.1) using modified Euler's method with h=0.05

	x	y	
x_0	0	1	y_0
x_1	0.05	1.0531	y_1
x_2	0.1	1.1055	y_2

Sol) Given $f(x,y) = x^2 + y, x_0 = 0, y_0 = 1, h = 0.05\ and\ f(x_0, y_0) = 1$

Euler's formula gives $y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.05(1) = 1.05$

From the modified Euler's formula,

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(n)}\right)\right]\ where\ n = 0,1,2,\ldots$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right] = 1 + \frac{0.05}{2}[1 + (0.05)^2 + 1.05] = 1.0513125$$

$y_1{}^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1{}^{(1)}\right)\right] = 1 + \frac{0.05}{2}[1 + (0.05)^2 + 1.0513125] = 1.05134531$

Since the values of $y_1{}^{(1)}, y_1{}^{(2)}$ are equal, we take $y_1 = y(0.05) = 1.0513$

Now with $x_1 = 0.05, y_1 = 1.0513\ and\ taking\ h = 0.05$

$y_2{}^{(n+1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2{}^{(n)}\right)\right]\ where\ n = 0,1,2, \ldots.$

We have $f(x_1, y_1) = (0.05)^2 + 1.0513 = 1.0538$

$y_2{}^{(0)} = y_1 + hf(x_1, y_1) = 1.0513 + 0.05(1.0538) = 1.104$

$y_2{}^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2{}^{(0)}\right)\right] = 1.0513 + \frac{0.05}{2}[1.0538 + (0.1)^2 + 1.104] = 1.1055$

$y_2{}^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2{}^{(1)}\right)\right] = 1.0513 + \frac{0.05}{2}[1.0538 + (0.1)^2 + 1.1055] = 1.1055$

Since the values of $y_2{}^{(1)}, y_2{}^{(2)}$ are equal, we take $y_2 = y(0.1) = 1.1055$

3. Solve $y^1 = x^2 + y\ with\ y(0) = 1$ and find y (0.02) using modified Euler's method with h=0.01

Sol) Given $f(x, y) = x^2 + y\ , x_0 = 0, y_0 = 1\ , h = 0.01\ and\ f(x_0, y_0) = 1$

Euler's formula gives $y_1{}^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.01(1) = 1.01$

From the modified Euler's formula,

x		y	
x_0	0	1	y_0
x_1	0.01	1.0101	y_1
x_2	0.02	1.0204	y_2

$y_1{}^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1{}^{(n)}\right)\right]\ where\ n = 0,1,2, \ldots$

$y_1{}^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1{}^{(0)}\right)\right] = 1 + \frac{0.01}{2}[1 + (0.01)^2 + 1.01] = 1.0100505$

$y_1{}^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1{}^{(1)}\right)\right] = 1 + \frac{0.01}{2}[1 + (0.01)^2 + 1.0100505] = 1.01010025$

Since the values of $y_1{}^{(1)}, y_1{}^{(2)}$ are equal, we take $y_1 = y(0.05) = 1.0101$

Now with $x_1 = 0.01, y_1 = 1.0101\ and\ taking\ h = 0.01$

$y_2{}^{(n+1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2{}^{(n)}\right)\right]\ where\ n = 0,1,2, \ldots.$

We have $f(x_1, y_1) = (0.01)^2 + 1.0101 = 1.0102$

$y_2{}^{(0)} = y_1 + hf(x_1, y_1) = 1.0102 + 0.01(1.0102) = 1.02$

$y_2{}^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2{}^{(0)}\right)\right] = 1.0102 + \frac{0.01}{2}[1.0102 + (0.02)^2 + 1.02] = 1.020353$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)\right] = 1.0102 + \frac{0.01}{2}[1.0102 + (0.02)^2 + 1.020353] = 1.02035476$$

Since the values of $y_2^{(1)}, y_2^{(2)}$ are equal, we take $y_2 = y(0.02) = 1.0204$

4. Solve $y^1 = x^2 + y$ *with* $y(0) = 1$ and find y (0.04) using modified Euler's method with h=0.02

Sol) Given $f(x, y) = x^2 + y\,, x_0 = 0, y_0 = 1\,, h = 0.02$ *and* $f(x_0, y_0) = 1$

Euler's formula gives $y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.02(1) = 1.02$

x		y	
x_0	0	1	y_0
x_1	0.02	1.0202	y_1
x_2	0.04	1.0408	y_2

From the modified Euler's formula,

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(n)}\right)\right] \text{ where } n = 0,1,2, \ldots$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right] = 1 + \frac{0.02}{2}[1 + (0.02)^2 + 1.02] = 1.0202$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right] = 1 + \frac{0.02}{2}[1 + (0.02)^2 + 1.0202] = 1.0202$$

Since the values of $y_1^{(1)}, y_1^{(2)}$ are equal, we take $y_1 = y(0.05) = 1.0202$

Now with $x_1 = 0.02, y_1 = 1.0202$ *and taking* $h = 0.02$

$$y_2^{(n+1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(n)}\right)\right] \text{ where } n = 0,1,2, \ldots.$$

We have $f(x_1, y_1) = (0.02)^2 + 1.0202 = 1.0206$

$$y_2^{(0)} = y_1 + hf(x_1, y_1) = 1.0202 + 0.02(1.0206) = 1.0406$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(0)}\right)\right] = 1.0202 + \frac{0.02}{2}[1.0202 + (0.04)^2 + 1.0406] = 1.0408$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)\right] = 1.0202 + \frac{0.02}{2}[1.0202 + (0.04)^2 + 1.0408] = 1.0408$$

Since the values of $y_2^{(1)}, y_2^{(2)}$ are equal, we take $y_2 = y(0.04) = 1.0408$

5. Solve $y^1 = x + y$ *with* $y(0) = 1$ using modified Euler's method .Show that y (0.05)=1.052564

Sol) Given $f(x, y) = x + y\,, x_0 = 0, y_0 = 1\,, h = 0.05$ *and* $f(x_0, y_0) = 1$

Euler's formula gives $y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.05(1) = 1.05$

From the modified Euler's formula,

x		y	
x_0	0	1	y_0
x_1	0.05	0.052564	y_1

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(n)}\right)\right] \text{ where } n = 0,1,2, \ldots$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right] = 1 + \frac{0.05}{2}[1 + 0.05 + 1.05] = 1.0525$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right] = 1 + \frac{0.05}{2}[1 + 0.05 + 1.0525] = 1.0525625$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right] = 1 + \frac{0.05}{2}[1 + 0.05 + 1.0525625] = 1.052564$$

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(3)}\right)\right] = 1 + \frac{0.05}{2}[1 + 0.05 + 1.052564] = 1.052564$$

Since the values of $y_1^{(3)}, y_1^{(4)}$ are equal, we take $y_1 = y(0.05) = 1.052564$

6. Solve $y^1 = \log(x + y)$ *with* $y(0) = 1$ and find y(0.2) using modified Euler's method .

Sol) Given $f(x, y) = \log(x + y)\,, x_0 = 0, y_0 = 1\,, h = 0.2$ *and* $f(x_0, y_0) = 0$

	x	y	
x_0	0	1	y_0
x_1	0.2	1.0082	y_1

Euler's formula gives $y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.2(0) = 1$

From the modified Euler's formula,

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(n)}\right)\right] \quad where\ n = 0,1,2, \ldots$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right] = 1 + \frac{0.2}{2}[0 + \log(0.2 + 1)] = 1 + \frac{0.2}{2}(0.0792) = 1.00792$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right] = 1 + \frac{0.2}{2}[\log(0.2 + 0.00792)] = 1.0082$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right] = 1 + \frac{0.2}{2}[\log(0.2 + 1.0082)] = 1.0082$$

Since the values of $y_1^{(2)}, y_1^{(3)}$ are equal, we take $y_1 = y(0.2) = 1.0082$

Runge-Kutta method of second order:

1. Solve $y^1 = x^2 + y$ *with* $y(0) = 1$ and find y (0.1) using R.K second order method with h=0.05

Sol) Given $f(x, y) = x^2 + y\,, x_0 = 0, y_0 = 1$ *and taking* $h = 0.05$

Runge-Kutta method is $y_1 = y_0 + \frac{1}{2}[k_1 + k_2]$

where $k_1 = hf(x_0, y_0)$ *and* $k_2 = hf(x_0 + h, y_0 + k_1)$

	x	y	
x_0	0	1	y_0
x_1	0.05	1.0751	y_1
x_2	0.1	1.1561	y_2

$$k_1 = hf(x_0, y_0) = h({x_0}^2 + y_0) = 0.05(0 + 1) = 0.05$$

$$k_2 = h\,f(x_0 + h, y_0 + k_1) = hf(0.05,1 + 1) = 0.05[(0.05)^2 + 2] = 0.100125$$

$$y_1 = y_0 + \frac{1}{2}[k_1 + k_2] = 1 + \frac{1}{2}[0.05 + 0.100125] = 1.0751$$

Next take $x_1 = 0.05, y_1 = 1.0751$ $\quad y_1 = y_0 + \frac{1}{2}[k_1 + k_2]$

$k_1 = hf(x_1, y_1) = h({x_1}^2 + y_1) = 0.05(0.0025 + 1.0751) = 0.05388$

$k_2 = h\,f(x_1 + h, y_1 + k_1) = hf(0.1, 1.0751 + 1.0776) = 0.05[(0.1)^2 + 2.1527] = 0.108135$

$$y_2 = y_1 + \frac{1}{2}[k_1 + k_2] = 1.0751 + \frac{1}{2}[0.05388 + 0.108135] = 1.1561$$

Hence $y_2 = y(0.1) = 1.1561$

2. Solve $y^1 = x^2 + y\ with\ y(0) = 1$ and find y(0.02) y(0.04) using R.K second order with h=0.02

Sol) Given $f(x, y) = x^2 + y\,, x_0 = 0, y_0 = 1\ and\ taking\ h = 0.02$

x		y	
x_0	0	1	y_0
x_1	0.02	1.03	y_1
x_2	0.04	1.0609	y_2

Runge-Kutta method is $y_1 = y_0 + \frac{1}{2}[k_1 + k_2]$

$where\ \ k_1 = hf(x_0, y_0) \quad and \quad k_2 = h\,f(x_0 + h, y_0 + k_1)$

$k_1 = hf(x_0, y_0) = h({x_0}^2 + y_0) = 0.02(0 + 1) = 0.02$

$k_2 = h\,f(x_0 + h, y_0 + k_1) = hf(0.02, 1 + 1) = 0.02[(0.02)^2 + 2] = 0.040008$

$$y_1 = y_0 + \frac{1}{2}[k_1 + k_2] = 1 + \frac{1}{2}[0.02 + 0.040008] = 1.03$$

Next take $x_1 = 0.02, y_1 = 1.03$ $\quad y_1 = y_0 + \frac{h}{2}[k_1 + k_2]$

$k_1 = hf(x_1, y_1) = h({x_1}^2 + y_1) = 0.02(0.0004 + 1.03) = 0.020608$

$k_2 = hf(x_1 + h, y_1 + k_1) = hf(0.04, 1.03 + 1.0304) = 0.02[(0.04)^2 + 2.0604] = 0.04124$

$$y_2 = y_1 + \frac{1}{2}[k_1 + k_2] = 1.03 + \frac{1}{2}[0.020608 + 0.04124] = 1.0609$$

Hence $y_2 = y(0.04) = 1.0609$

3. Solve $y^1 = y - x\ with\ y(0) = 2$ and find y (0.1) y (0.2) using R.K second order with h=0.1

Sol) Given $f(x, y) = y - x\,, x_0 = 0, y_0 = 2\ and\ taking\ h = 0.1$

x		y	
x_0	0	2	y_0
x_1	0.1	2.205	y_1
x_2	0.2	2.4210	y_2

Runge-Kutta method is $y_1 = y_0 + \frac{1}{2}[k_1 + k_2]$

$where\ \ k_1 = hf(x_0, y_0) \quad and \quad k_2 = hf(x_0 + h, y_0 + k_1)$

$k_1 = hf(x_0, y_0) = h(y_0 - x_{0)} = 0.1(2 - 0) = 0.2$

$k_2 = h\,f(x_0 + h, y_0 + k_1) = hf(0.1, 2 + 0.2) = 0.1(2.2 - 0.1) = 0.21$

$y_1 = y_0 + \frac{1}{2}[k_1 + k_2] = 2 + \frac{1}{2}[0.2 + 0.21] = 2.205$

Next take $x_1 = 0.1, y_1 = 2.2050 \quad y_2 = y_1 + \frac{1}{2}[k_1 + k_2]$

$k_1 = hf(x_1, y_1) = h(y_1 - x_1) = 0.1(2.205 - 0.1) = 0.2105$

$k_2 = hf(x_1 + h, y_1 + k_1) = hf(0.2, 2.205 + 0.2105) = 0.1[(2.4155 - 0.2] = 0.22155$

$y_2 = y_1 + \frac{1}{2}[k_1 + k_2] = 2.2050 + \frac{1}{2}[0.2105 + 0.22155] = 2.4210$

Hence $y_2 = y(0.2) = 2.4210$

4. Solve $y^1 = x^2 + y^2$ *with* $y(1) = 1.5$ and find y (1.2) using R.K second order with h=0.1

Sol) Given $f(x, y) = x^2 + y^2, x_0 = 1, y_0 = 1.5$ *and taking* $h = 0.1$

x		y	
x_0	1	1.5	y_0
x_1	1.1	1.8890	y_1
x_2	1.2	2.4800	y_2

Runge-Kutta method is $y_1 = y_0 + \frac{1}{2}[k_1 + k_2]$

where $k_1 = hf(x_0, y_0)$ *and* $k_2 = hf(x_0 + h, y_0 + k_1)$

$k_1 = hf(x_0, y_0) = h({x_0}^2 + {y_0}^2) = 0.1(1 + 2.25) = 0.325$

$k_2 = h\, f(x_0 + h, y_0 + k_1) = hf(1.1, 1.5 + 0.325) = 0.1((1.1)^2 + (1.825)^2) = 0.4540625$

$y_1 = y_0 + \frac{1}{2}[k_1 + k_2] = 1.5 + \frac{1}{2}[0.325 + 0.4540625] = 1.8890$

Next take $x_1 = 1.1, y_1 = 1.8890 \quad y_2 = y_1 + \frac{1}{2}[k_1 + k_2]$

$k_1 = hf(x_1, y_1) = h({x_1}^2 + {y_1}^2) = 0.1[(1.1)^2 + (1.889)^2] = 0.1(1.21 + 3.568) = 0.4778321$

$k_2 = hf(x_1 + h, y_1 + k_1) = hf(1.2, 1.8890 + 0.4778321) = 0.1[((1.2)^2 + (2.3668321)^2] = 0.7042$

$y_2 = y_1 + \frac{1}{2}[k_1 + k_2] = 1.8890 + \frac{1}{2}[0.4778 + 0.7042] = 2.4800$

Hence $y_2 = y(0.2) = 2.4800$

5. Using R.K method solve $10y^1 = x^2 + y^2$ *with* $y(0) = 1$ in the interval $0 < x \leq 0.4$ *with* $h = 0.2$

Ans: By using second order $y_1 = y(0.2) = 1.0208$ and $y_2 = y(0.4) = 1.0441$

Runge-Kutta method of fourth order:

1. Solve $y^1 = y - x \; with \; y(0) = 2$ and find y (0.1) y (0.2) using R.K fourth order method

Sol) Given $f(x, y) = y - x \,, x_0 = 0, y_0 = 2 \; and \; taking \; h = 0.1$

Fourth order Runge-Kutta method is $y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

x		y	
x_0	0	2	y_0
x_1	0.1	2.2052	y_1
x_2	0.2	2.4214	y_2

$where \;\; k_1 = hf(x_0, y_0), \;\; and \quad k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right),$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right), k_4 = hf(x_0 + h, y_0 + \; k_3)$$

$$k_1 = hf(x_0, y_0) = (0.1)(2 - 0) = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)f(0 + \frac{1}{2}(0.1), 2 + \frac{1}{2}(0.2) = (0.1)[2.10 - 0.05] = 0.1(2.05) = 0.205$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)f(0.05, 2 + \frac{1}{2}(0.205) = (0.1)[2.1025 - 0.05] = 0.20525$$

$$k_4 = hf(x_0 + h, y_0 + \; k_3) = (0.1)f(0.1, 2 + 0.20525) = (0.1)[2.20525 - 0.1] = 0.21053$$

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 2 + \frac{1}{6}[0.2 + 2(0.205) + 2(0.20525) + 0.21053] = 2.2052$$

Next take $x_1 = 0.1, y_1 = 2.2052 \quad f(x, y) = y - x$ and $taking \; h = 0.1$

$$k_1 = hf(x_1, y_1) = (0.1)(2.2052 - 0.1) = 0.2105$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1)f(0.1 + \frac{1}{2}(0.1), 2.2052 + \frac{1}{2}(0.2105))$$
$$= (0.1)\left[2.2052 + \frac{1}{2}(0.2105) - 0.15\right] = 0.1(2.16046) = 0.2160$$

$$k_3 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.1)f(0.15, 2.2052 + \frac{1}{2}(0.216) = (0.1)\left[2.2052 + \frac{1}{2}(0.216) - 0.15\right] =$$

$$k_4 = hf(x_1 + h, y_1 + \; k_3) = (0.1)f(0.2, 2.2052 + 0.2163) = (0.1)[2.4215 - 0.2] = 0.2222$$

$$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 2.2052 + \frac{1}{6}[0.2105 + 2(0.216) + 2(0.2163) + 0.2222] = 2.4214$$

Hence $y_1 = y(0.1) = 2.2052$ and $y_2 = y(0.2) = 2.4214$

2. Solve $y^1 = x^2 + y^2 \; with \; y(0) = 1$ and find y (0.2) using R.K fourth order with h=0.1

Sol) Given $f(x, y) = x^2 + y^2 \,, x_0 = 0, y_0 = 1 and \; taking \; h = 0.1$

Fourth order Runge-Kutta method is $y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$where\ \ k_1 = hf(x_0, y_0),\ and\ \ \ k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right),$

$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right), k_4 = hf(x_0 + h, y_0 + \ k_3)$

	x	y	
x_0	0	1	y_0
x_1	0.1	1.1115	y_1
x_2	0.2	1.2530	y_2

$k_1 = hf(x_0, y_0) = (0.1)(0^2 + 1^2) = 0.1$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)f(0 + \frac{1}{2}(0.1), 1 + \frac{1}{2}(0.1) = (0.1)[(0.05)^2 + (1.05)^2] = 0.1105$

$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)f(0.05, 1 + \frac{1}{2}(0.1105) = (0.1)[(0.05)^2 + (1.05525)^2] = 0.1116$

$k_4 = hf(x_0 + h, y_0 + \ k_3) = (0.1)f(0.1, 1 + 0.1116) = (0.1)[(0.1)^2 + (1.1116)^2] = 0.1246$

$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 1 + \frac{1}{6}[0.1 + 2(0.1105) + 2(0.1116) + 0.1245] = 1.1115$

Next take $x_1 = 0.1, y_1 = 1.1115 \quad f(x, y) = x^2 + y^2$ and $taking\ h = 0.1$

$k_1 = hf(x_1, y_1) = (0.1)[(0.1)^2 + (1.1115)^2] = 0.1245$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1)f(0.1 + \frac{1}{2}(0.1), 1.1115 + \frac{1}{2}(0.1245)) = (0.1)[(0.15)^2 + (1.17375)^2]$
$= 0.1(1.400189) = 0.1400$

$k_3 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.1)f(0.15, 1.1115 + \frac{1}{2}(0.14) = (0.1)[(0.15)^2 + (1.1815)^2] = 0.1418$

$k_4 = hf(x_1 + h, y_1 + \ k_3) = (0.1)f(0.2, 1.1115 + 0.1418) = (0.1)[(0.2)^2 + (1.2533)^2] = 0.1611$

$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 1.1115 + \frac{1}{6}[0.1245 + 2(0.14) + 2(0.1418) + 0.1611] = 1.2530$

Hence $y_1 = y(0.1) = 1.1115$ and $y_2 = y(0.2) = 1.2530$

3. Solve $y^1 = x^2 + y^2\ with\ y(1) = 1.5$ and find y (1.2) using R.K fourth order with h=0.1

Ans: $y_1 = y(1.1) = 1.8955$ and $y_2 = y(1.2) = 2.5043$

4. Using R.K fourth order method solve $10y^1 = x^2 + y^2\ with\ y(0) = 1$ in the interval $0 < x \leq 0.2\ with\ h = 0.1$

Sol) Given $f(x, y) = \frac{x^2 + y^2}{10}, x_0 = 0, y_0 = 1 and\ taking\ h = 0.1$

Fourth order Runge-Kutta method is $y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

	x	y	
x_0	0	1	y_0
x_1	0.1	1.0101	y_1
x_2	0.2	1.0206	y_2

$where\ \ k_1 = hf(x_0, y_0),\ and\ \ \ k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right),$

$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$, $k_4 = hf(x_0 + h, y_0 + \ k_3)$

$k_1 = hf(x_0, y_0) = \frac{(0.1)(0^2+1^2)}{10} = 0.01$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)f(0 + \frac{1}{2}(0.1), 1 + \frac{1}{2}(0.01) = \frac{(0.1)}{10}[(0.05)^2 + (1.005)^2] = 0.0101$

$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)f(0.05, 1 + \frac{1}{2}(0.0101) = \frac{(0.1)}{10}[(0.05)^2 + (1.0051)^2] = 0.0101$

$k_4 = hf(x_0 + h, y_0 + \ k_3) = (0.1)f(0.1, 1 + 0.0101) = \frac{(0.1)}{10}[(0.1)^2 + (1.0101)^2] = 0.0103$

$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 1 + \frac{1}{6}[0.01 + 2(0.0101) + 2(0.0101) + 0.0103] = 1.0101$

Next take $x_1 = 0.1, y_1 = 1.0101$ $\quad f(x,y) = \frac{(x^2+y^2)}{10}$ and $taking\ h = 0.1$

$k_1 = hf(x_1, y_1) = \frac{(0.1)}{10}[(0.1)^2 + (1.0101)^2] = 0.0103$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1)f(0.1 + \frac{1}{2}(0.1), 1.0101 + \frac{1}{2}(0.0103)) = \frac{(0.1)}{10}[(0.15)^2 + (1.0153)^2]$
$= 0.0105$

$k_3 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.1)f(0.15, 1.0101 + \frac{1}{2}(0.0105) = \frac{(0.1)}{10}[(0.15)^2 + (1.0154)^2] = 0.0105$

$k_4 = hf(x_1 + h, y_1 + \ k_3) = (0.1)f(0.2, 1.0101 + 0.0105) = \frac{(0.1)}{10}[(0.2)^2 + (1.0206)^2] = 0.0108$

$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 1.0101 + \frac{1}{6}[0.0103 + 2(0.0105) + 2(0.0105) + 0.0108] = 1.0206$

Hence $y_1 = y(0.1) = 1.0101$ and $y_2 = y(0.2) = 1.0206$

5. Using R.K fourth order method find y (0.4) for the equation $\frac{dy}{dx} = \frac{y-x}{y+x}$ $with\ y(0) = 1\ take\ \ h = 0.2$

Sol) Given $f(x,y) = \frac{y-x}{y+x},\ x_0 = 0, y_0 = 1\ \ taking\ \ h = 0.2$

Fourth order Runge-Kutta method is $y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$where\ \ k_1 = hf(x_0, y_0),\ and\ \ \ k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right),$

$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$, $k_4 = hf(x_0 + h, y_0 + \ k_3)$

	x	y	
x_0	0	1	y_0
x_1	0.2	1.1561	y_1
x_2	0.4	1.2778	y_2

$k_1 = hf(x_0, y_0) = (0.2)[\frac{1-0}{1+0}] = 0.2$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.2)f(0 + \frac{1}{2}(0.2), 1 + \frac{1}{2}(0.2) = (0.2)[\frac{1.1 - 0.1}{1.1 + 0.1}] = 0.16666$

$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.2)f(0.1, 1 + \frac{1}{2}(0.16666) = (0.2)[\frac{1.08333 - 0.1}{1.08333 + 0.1}] = 0.16619$

$k_4 = hf(x_0 + h, y_0 + k_3) = (0.2)f(0.2, 1 + 0.16619) = (0.2)[\frac{1.16619 - 0.2}{1.16619 + 0.2}] = 0.07072$

$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 1 + \frac{1}{6}[0.2 + 2(0.16666) + 2(0.16619) + 0.07072] = 1.15607$

Next take $x_1 = 0.2, y_1 = 1.15607$ $\quad f(x, y) = \frac{(y-x)}{y+x}$ and $taking\ h = 0.2$

$k_1 = hf(x_1, y_1) = (0.2)[\frac{1.15607 - 0.2}{1.15607 + 0.2}] = 0.141$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.2)f(0.2 + \frac{1}{2}(0.2), 1.15607 + \frac{1}{2}(0.141)) = (0.2)\left[\frac{1.22657 - 0.3}{1.22657 + 0.3}\right]$
$= 0.1214$

$k_3 = h\, f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.2)f(0.3, 1.15607 + \frac{1}{2}(0.1214) = (0.2)\left[\frac{1.21677 - 0.3}{1.21677 + 0.3}\right] = 0.12088$

$k_4 = hf(x_1 + h, y_1 + k_3) = (0.2)f(0.4, 1.15607 + 0.12088) = (0.2)\left[\frac{1.27695 - 0.4}{1.27695 + 0.4}\right] = 0.1046$

$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 1.15607 + \frac{1}{6}[0.141 + 2(0.1214) + 2(0.12088) + 0.1046] = 1.2778$

Hence $y_1 = y(0.2) = 1.1561$ and $y_2 = y(0.4) = 1.2778$

6. Using R.K .method to approximate y when x=0.1 given that x=0 when y=1 and $\frac{dy}{dx} = 3x + \frac{1}{2}y\ and\ find\ y(0.2)$

Sol) Given $f(x, y) = 3x + \frac{1}{2}y \quad x_0 = 0, y_0 = 1\ and\ taking\ h = 0.1$

x		y	
x_0	0	1	y_0
x_1	0.1	1.0582	y_1
x_2	0.2	1.1585	y_2

Fourth order Runge-Kutta method is $y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$where\ \ k_1 = hf(x_0, y_0),\ and\ \ k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right),$

$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right), k_4 = hf(x_0 + h, y_0 + k_3)$

$k_1 = hf(x_0, y_0) = (0.1)[3(0) + \frac{1}{2}(1)] = 0.05$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)f(0 + \frac{1}{2}(0.1), 1 + \frac{1}{2}(0.05) = (0.1)\left[3(0.05) + \frac{1}{2}(1.025)\right] = 0.06625$$

$$k_3 = h\ f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)f(0.05, 1 + \frac{1}{2}(0.06625) = (0.1)\left[3(0.05) + \frac{1}{2}(1.033125)\right]$$
$$= 0.06665625$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)f(0.1, 1 + 0.06665625) = (0.1)\left[3(0.1) + \frac{1}{2}(1.06665625)\right] = 0.003333$$

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 1 + \frac{1}{6}[0.05 + 2(0.06625) + 2(0.06665625) + 0.003333] = 1.05819$$

Next take $x_1 = 0.1, y_1 = 1.05819$ $\quad f(x, y) = 3x + \frac{1}{2}y$ and $taking\ h = 0.1$

$$k_1 = hf(x_1, y_1) = (0.1)[3(0.1) + \frac{1}{2}(1.05819)] = 0.08291$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1)f(0.1 + \frac{1}{2}(0.1), 1.05819 + \frac{1}{2}(0.08291))$$
$$= (0.1)[3(0.15) + \frac{1}{2}(1.09965)] = 0.09998$$

$$k_3 = h\ f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.1)f(0.15, 1.05819 + \frac{1}{2}(0.09998) = (0.1)\left[3(0.15) + \frac{1}{2}(1.10818)\right]$$
$$= 0.10041$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1)f(0.2, 1.05819 + 0.10041) = (0.1)\left[3(0.2) + \frac{1}{2}(1.1586)\right] = 0.11793$$

$$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 1.05819 + \frac{1}{6}[0.08291 + 2(0.09998) + 2(0.10041) + 0.11793]$$
$$= 1.1585$$

Hence $y_1 = y(0.1) = 1.0582$ and $y_2 = y(0.2) = 1.1585$

7. Solve $y^1 = x + y\ with\ y(0) = 1$ and find y (0.1) y (0.2) using R.K fourth order method

Sol) Given $f(x, y) = x + y$, $x_0 = 0, y_0 = 1\ and\ taking\ h = 0.1$

Fourth order Runge-Kutta method is $y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

	x	y	
x_0	0	1	y_0
x_1	0.1	1.1103	y_1
x_2	0.2	1.2428	y_2

$where\ k_1 = hf(x_0, y_0),\ and\quad k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right),$

$$k_3 = h\ f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right), k_4 = hf(x_0 + h, y_0 + k_3)$$

$$k_1 = hf(x_0, y_0) = (0.1)(0 + 1) = 0.1$$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)f(0 + \frac{1}{2}(0.1), 1 + \frac{1}{2}(0.1) = (0.1)[0.05 + 1.05] = 0.1(1.1) = 0.11$

$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)f(0.05, 1 + \frac{1}{2}(0.11) = (0.1)[0.05 + 1.055] = 0.1105$

$k_4 = hf(x_0 + h, y_0 + \ k_3) = (0.1)f(0.1, 1 + 0.1105) = (0.1)[0.1 + 1.1105] = 0.12105$

$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 1 + \frac{1}{6}[0.1 + 2(0.11) + 2(0.1105) + 0.12105] = 1.11034$

Next take $x_1 = 0.1, y_1 = 1.11034 \quad f(x,y) = x + x$ and $taking\ h = 0.1$

$k_1 = hf(x_1, y_1) = (0.1)(0.1 + 1.11034) = 0.121034$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1)f(0.1 + \frac{1}{2}(0.1), 1.11034 + \frac{1}{2}(0.121034)) = (0.1)[0.15 + 1.170857]$
$= 0.1(1.320857) = 0.1320857$

$k_3 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.1)f(0.15, 1.11034 + \frac{1}{2}(0.1320857) = (0.1)[0.15 + 1.17638285]$
$= 0.1326382$

$k_4 = hf(x_1 + h, y_1 + \ k_3) = (0.1)f(0.2, 1.11034 + 0.1326382) = (0.1)[0.2 + 1.2429783] = 0.1442978$

$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 1.11034 + \frac{1}{6}[0.121034 + 2(0.1320857) + 2(0.326382) + 0.1442978] = 1.242803$

Hence $y_1 = y(0.1) = 1.1103$ and $y_2 = y(0.2) = 1.2428$

8. Explain R.K method and use it to solve $y^1 = xy\ for\ x = 1.4$ given that $x = 1, y = 2.$

Sol) Given $f(x,y) = xy\ , x_0 = 1, y_0 = 2\ and\ taking\ h = 0.2$

Fourth order Runge-Kutta method is $y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

x		y	
x_0	1	2	y_0
x_1	1.2	2.4921	y_1
x_2	1.4	3.2321	y_2

$where\ \ k_1 = hf(x_0, y_0),\ and \quad k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right),$

$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right), k_4 = hf(x_0 + h, y_0 + \ k_3)$

$k_1 = hf(x_0, y_0) = (0.2)(2) = 0.4$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.2)f(1 + \frac{1}{2}(0.2), 2 + \frac{1}{2}(0.4) = (0.2)[(1.1)(2.2)] = 0.484$

$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.2)f(1.1, 2 + \frac{1}{2}(0.484) = (0.2)[(1.1)(2.242)] = 0.4932$

$k_4 = hf(x_0 + h, y_0 + k_3) = (0.2)f(1.2, 2 + 0.4932) = (0.2)[(1.2)(2.4932)] = 0.5984$

$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 2 + \frac{1}{6}[0.4 + 2(0.484) + 2(0.4932) + 0.5984] = 2.4921$

Next take $x_1 = 1.2, y_1 = 2.4921$ $f(x, y) = xy$ and $taking\ h = 0.2$

$k_1 = hf(x_1, y_1) = (0.2)(1.2)(2.4921) = 0.5981$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.2)f(1.2 + \frac{1}{2}(0.2), 2.4921 + \frac{1}{2}(0.5981)) = (0.2)[(1.3)(2.79115)] = 0.7257$$

$$k_3 = h\ f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.2)f(1.3, 2.4921 + \frac{1}{2}(0.7257) = (0.2)[(1.3)(2.85495)] = 0.7423$$

$k_4 = hf(x_1 + h, y_1 + k_3) = (0.2)f(1.4, 2.4921 + 0.7423) = (0.2)[(1.4)(3.2344)] = 0.9056$

$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 2.4921 + \frac{1}{6}[0.5981 + 2(0.7257) + 2(0.7423) + 0.9056] = 3.2321$

Hence $y_1 = y(1.2) = 2.4921$ and $y_2 = y(1.4) = 3.2321$

9. Apply the fourth order R.K method to find y (0.1) and y (0.2), given $y^1 = xy + y^2, y(0) = 1$

Ans) $y_1 = y(0.1) = 1.1133$ and $y_2 = y(0.2) = 1.2728$

10. Find y (0.1) and y (0.2) using R.K fourth order method, given that $y^1 = x^2 - y\ and\ y(0) = 1$

Ans) $y_1 = y(0.1) = 0.9052$ and $y_2 = y(0.2) = 0.8213$